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HIGH-FREQUENCY RESOLVENT ESTIMATES ON ASYMPTOTICALLY EUCLIDEAN WARPED PRODUCTS

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Abstract. We consider the resolvent on asymptotically Euclidean warped product manifolds in an appropriate 0-Gevrey class, with trapped sets consisting of only finitely many components. We prove that the high-frequency resolvent is either bounded by $C_\epsilon|\lambda|^\epsilon$ for any $\epsilon > 0$, or blows up faster than any polynomial (at least along a subsequence). A stronger result holds if the manifold is analytic. The method of proof is to exploit the warped product structure to separate variables, obtaining a one-dimensional semiclassical Schrödinger operator. We then classify the microlocal resolvent behaviour associated to every possible type of critical value of the potential, and translate this into the associated resolvent estimates. Weakly stable trapping admits highly concentrated quasimodes and fast growth of the resolvent. Conversely, using a delicate inhomogeneous blowup procedure loosely based on the classical positive commutator argument, we show that any weakly unstable trapping forces at least some spreading of quasimodes.

As a first application, we conclude that either there is a resonance free region of size $|\operatorname{Im}\lambda| \leqslant C_\epsilon|\operatorname{Re}\lambda|^{-1-\epsilon}$ for any $\epsilon > 0$, or there is a sequence of resonances converging to the real axis faster than any polynomial. Again, a stronger result holds if the manifold is analytic. As a second application, we prove a spreading result for weak quasimodes in partially rectangular billiards.

1. Introduction

In this paper, we consider manifolds which have a warped product structure and are asymptotically Euclidean with a certain 0-Gevrey regularity. Our main result is that the cutoff resolvent is either (almost) bounded or blows up faster than any polynomial. Of course, the proof gives much more information than this simple statement, but for aesthetic reasons we prefer to phrase it in this fashion. Let us state the main result.

Theorem 1. *Let X be a 0-Gevrey smooth $\mathcal{G}^0_\tau$, $\tau < \infty$, warped product manifold without boundary which is a short range perturbation of Euclidean space (with one or two infinite ends). Assume also that the trapped set on X has finitely many connected components. Let $-\Delta$ be the Laplace-Beltrami operator on X.*

Then either

1: *For every $\varphi \in \mathcal{C}^\infty_c(X)$ and every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\|\varphi(-\Delta - (\lambda - i0)^2)^{-1}\varphi\| \leqslant C_\epsilon|\lambda|^\epsilon, \ \ |\lambda| \gg 1. \tag{1.1}$$

or

2: *For every $N > 0$, there exists $\varphi \in \mathcal{C}^\infty_c(X)$, $C_N > 0$, and a sequence $\lambda_j \in \mathbb{R}$, $\lambda_j \to \infty$, such that*

$$\|\varphi(-\Delta - (\lambda_j - i0)^2)^{-1}\varphi\| \geqslant C_N|\lambda_j|^N. \tag{1.2}$$

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Remark 1.1. The warped product structure at infinity can be replaced by a number of different non-trapping infinite "ends", using the recent gluing theorem of Datchev-Vasy [DV12] (see also [Chr08] and Appendix A).

The dichotomy in Theorem 1 is from the following idea: if there is any weakly stable trapping on X, then there are well-localized quasimodes, and we are in Case 2 of the Theorem. If all the trapping is at least weakly unstable, we need to prove there is weak microlocal non-concentration near each connected component of the trapped set, as well as prove that there is no strong tunneling between different connected components (i.e. that the different connected components at the same energy don't "talk" to each other too much).

As we shall see, the worst behaviour in (1.1) in Theorem 1 comes from weakly unstable trapping which is infinitely degenerate. Since such trapping cannot occur on an analytic manifold, there is a nice improvement in this case, given in the next Corollary.

Corollary 1.2. *In addition to the assumptions of Theorem 1, assume that the manifold X is analytic.*

Then either

1: *There exists $\delta > 0$ such that for every $\varphi \in \mathcal{C}_c^\infty(X)$, there is a constant $C > 0$ for which we have the estimate*

$$\|\varphi(-\Delta - (\lambda - i0)^2)^{-1}\varphi\| \leqslant C|\lambda|^{-\delta}, \ \ |\lambda| \to \infty. \tag{1.3}$$

or

2: *For every $N > 0$, there exists $\varphi \in \mathcal{C}_c^\infty(X)$, $C_N > 0$, and a sequence $\lambda_j \in \mathbb{R}$, $\lambda_j \to \infty$, such that*

$$\|\varphi(-\Delta - (\lambda_j - i0)^2)^{-1}\varphi\| \geqslant C_N|\lambda_j|^N. \tag{1.4}$$

Remark 1.3. Upon rescaling to a semiclassical problem, Theorem 1 states that the semiclassical cutoff resolvent is either controlled by $h^{-2-\epsilon}$ for any $\epsilon > 0$, or blows up faster than h^{-N} for any N. The corollary states that if the manifold is analytic, the first possibility can be replaced with $h^{-2+\delta}$ for some $\delta > 0$ fixed, depending on the trapping.

As usual, high energy resolvent estimates imply there are regions free of resonances by simple perturbation of the spectral parameter. On the other hand, the proof of the alternative large growth of the resolvent along a subsequence proceeds by quasimode construction. Then if our metric has a complex analytic extension outside of a compact set, we can apply the results of Tang-Zworski [TZ98] to conclude existence of resonances. This results in the following Corollary.

Corollary 1.4. *In addition to the assumptions of Theorem 1, assume X admits a complex analytic extension outside of a compact set so that resonances may be defined by complex scaling. Then either*

1: *For every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that the region*

$$\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant C_\epsilon|\operatorname{Re}\lambda|^{-1-\epsilon}, \ |\lambda| \gg 1\}$$

is free of resonances and the estimate (1.1) *holds there (with a suitably modified constant), or*

2: *For every $N > 0$, there exists a sequence of resonances $\{\lambda_j\}$ such that*

$$|\operatorname{Im}\lambda_j| \leqslant |\operatorname{Re}\lambda_j|^{-N}, \ |\lambda_j| \to \infty.$$

In particular, if X is analytic, then either there exists $\delta > 0$ and $C > 0$ such that the region

$$\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant C|\operatorname{Re}\lambda|^{-1+\delta},\ |\lambda| \gg 1\}$$

is free from resonances, or there is a sequence converging to the real axis at an arbitrarily fast polynomial rate.

1.1. **Resolvents and the local smoothing effect.** One of the many motivations for studying resolvents and resolvent estimates is to understand the local smoothing effect for the Schrödinger equation on manifolds with trapping. It is well known (see, for example, [Tao06, Doi96]) that on asymptotically Euclidean manifolds without trapping, solutions to the Schrödinger equation enjoy a $1/2$ derivative local smoothing effect. This says that, locally in space, and on average in time, solutions are $1/2$ derivative smoother than the initial data. To be precise, let X be such a manifold, $-\Delta$ the Laplacian on X, u_0 a Schwartz function on X, and $\chi \in \mathcal{C}_c^\infty(X)$ a cutoff function. Then the following estimate holds true for any $T > 0$:

$$\int_0^T \|\chi e^{it\Delta} u_0\|^2_{H^{1/2}(X)} dt \leqslant C_T \|u_0\|^2_{L^2(X)}.$$

There are several ways to prove such an estimate; one way proceeds through resolvent estimates (see Section 5 below). A nice benefit of using the resolvent formalism to understand local in time local smoothing (that is, for finite T) is that one really sees how the spectral estimates are related to the smoothing effect. Since one only needs a resolvent estimate in a fixed strip near the real axis, if one is in a situation where the limiting resolvent blows up, one simply uses the trivial bound away from the real axis to get a zero derivative smoothing effect (or just integrates the $L^2(X)$ mass in time). However, if the limiting resolvent has some decay, then there is a non-trivial local smoothing estimate. This is the case, for example, if the manifold is analytic and all of the trapping is at least weakly unstable. Let us state this as a corollary:

Corollary 1.5. *Let X be an analytic warped product manifold so that all of the assumptions of Corollary 1.2 hold. Assume also that every connected component of the trapped set is at least weakly unstable, so that conclusion* 1 *of Corollary 1.2 holds for some $\delta > 0$. Then for all $\chi \in \mathcal{C}_c^\infty(X)$, $u_0 \in \mathcal{S}(X)$, and $T > 0$, there exists $C_T > 0$ such that*

$$\int_0^T \|\chi e^{it\Delta} u_0\|^2_{H^{\delta/2}(X)} dt \leqslant C_T \|u_0\|^2_{L^2(X)}.$$

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2. Preliminaries

2.1. **The geometry.** We have assumed that our manifold X has a warped product structure with one or two infinite ends which are short range perturbations of $\mathbb{R}^n$. This means we are considering the manifold $X = \mathbb{R}_x \times \Omega_\theta^{n-1}$ (or $X = \mathbb{R}_x^+ \times \Omega_\theta^{n-1}$ if one infinite end), equipped with the metric

$$g = dx^2 + A^2(x) G_\theta,$$

where $A \in \mathcal{C}^\infty$ is a smooth function, $A \geqslant \epsilon > 0$ for some epsilon (or $A(x) = x$ for $x > 0$ near 0 if one infinite end), and G_θ is the metric on a smooth compact $n-1$ dimensional Riemannian manifold Ω^{n-1} without boundary. The short range assumption means that as $|x| \to \infty$, we have

$$|\partial^\alpha (g - g_E)| \leqslant C_\alpha \langle x \rangle^{-2-|\alpha|},$$

where

$$g_E = dx^2 + x^2 G_\theta.$$

This means that X is asymptotically Euclidean as $|x| \to \infty$. This assumption merely allows us to use standard techniques to glue resolvent estimates together without worrying about trapping at infinity. The assumptions can of course be weakened to "long-range" perturbation (following Vasy-Zworski [VZ00]), but this paper is really about the local phenomenon of trapping rather than having the most general "infinity". We use the notation $\theta \in \Omega^{n-1}$ to denote the "angular" directions. This is in analogue with the case of spherically symmetric warped product spaces where $\Omega^{n-1} = \mathbb{S}^{n-1}$ is the sphere and X is asymptotically $\mathbb{R}^n$.

From this metric, we get the volume form

$$d\mathrm{Vol} = A(x)^{n-1} dx d\sigma,$$

where σ is the volume measure on Ω^{n-1}. The Laplace-Beltrami operator acting on 0-forms is computed:

$$\Delta f = (\partial_x^2 + A^{-2}\Delta_{\Omega^{n-1}} + (n-1)A^{-1}A'\partial_x)f,$$

where $\Delta_{\Omega^{n-1}}$ is the (non-positive) Laplace-Beltrami operator on Ω^{n-1}.

We want to exploit the warped product structure to reduce spectral questions to a one-dimensional problem. Let us first conjugate to a problem on the flat cylinder. That is, let $Tu(x,\theta) = A^{(n-1)/2}(x)u(x,\theta)$ so that $\widetilde{\Delta} = T\Delta T^{-1}$ is essentially self-adjoint on $L^2(dxd\sigma)$, where σ is the usual volume measure on Ω^{n-1}. We have

$$-\widetilde{\Delta} = -\partial_x^2 - A^{-2}(x)\Delta_\Omega + V_1(x),$$

where

$$V_1(x) = \frac{n-1}{2}A''A^{-1} - \frac{(n-1)(n-3)}{4}(A')^2 A^{-2}.$$

Separating variables we write for $u \in L^2(dxd\sigma)$

$$u(x,\theta) = \sum_{l,k} u_{lk}(x)\varphi_{lk}(\theta),$$

where $\varphi_{lk}(\theta)$ are the eigenfunctions on Ω^{n-1} with eigenvalue λ_k^2. Then

$$-\widetilde{\Delta} u = \sum_{l,k} \varphi_{lk}(\theta) Q_k u_{lk},$$

where

$$Q_k \varphi(x) = (-\partial_x^2 + \lambda_k^2 A^{-2}(x) + V_1(x))\varphi(x).$$

Setting $h = \lambda_k^{-1}$ and rescaling, we end up with the semiclassical operator

$$P(h)\varphi(x) = (-h^2\partial_x^2 + V(x))\varphi(x),$$

where

$$V(x) = A^{-2}(x) + h^2 V_1(x).$$

We sometimes will write $V_0(x) = A^{-2}(x)$ for the principal part of the effective potential.

The semiclassical versions of Theorem 1 and Corollary 1.2 are given in the following.

Theorem 2. *Under the assumptions above, either*

1: *For every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ and every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$\|\varphi(-h^2\partial_x^2 + V - (z - i0))^{-1}\varphi\| \leqslant C_\epsilon h^{-2-\epsilon}, \;\; z \in I,$$

for a compact interval I, or

2: *For every $N > 0$, there exists $\varphi \in \mathcal{C}_c^\infty(X)$, $C_N > 0$, and $z \in \mathbb{R}$, $z \neq 0$, such that*

$$\|\varphi(-h^2\partial_x^2 + V - (z - i0))^{-1}\varphi\| \geqslant C_N h^{-N},$$

along a subsequence as $h \to 0+$.

Corollary 2.1. *In addition to the assumptions of Theorem 2, assume that the manifold X is analytic.*

1: *There exists $\delta > 0$ such that, for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ there is a constant $C > 0$ for which we have the following estimate*

$$\|\varphi(-h^2\partial_x^2 + V - (z - i0))^{-1}\varphi\| \leqslant C h^{-2+\delta}, \;\; z \in I,$$

for a compact interval I, or

2: *For any $N > 0$, there exists $\varphi \in \mathcal{C}_c^\infty(X)$, $C_N > 0$, and $z \in \mathbb{R}$, $z \neq 0$, such that*

$$\|\varphi(-h^2\partial_x^2 + V - (z - i0))^{-1}\varphi\| \geqslant C_N h^{-N}$$

along a subsequence as $h \to 0+$.

2.2. **The 0-Gevrey class $\mathcal{G}_\tau^0$.** Our manifolds already have very nice geometry as $|x| \to \infty$, and moreover we have separated variables. Since Ω^{n-1} is a $\mathcal{C}^\infty$ compact manifold without boundary, the only additional regularity assumptions we need to impose will be at the critical elements of the manifold X, that is, at the critical points of the function $A(x)$. In order to have a meaningful symbol class (especially once we are working with the calculus of 2 parameters), we need to know that near the critical elements, the function A is not too far away from being analytic. For this, we introduce the following 0-Gevrey classes of functions with respect to order of vanishing. For $0 \leqslant \tau < \infty$, let $\mathcal{G}_\tau^0(\mathbb{R})$ be the set of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ such that, for each $x_0 \in \mathbb{R}$, there exists a neighbourhood $U \ni x_0$ and a constant C such that, for all $0 \leqslant s \leqslant k$,

$$|\partial_x^k f(x) - \partial_x^k f(x_0)| \leqslant C(k!)^C |x - x_0|^{-\tau(k-s)} |\partial_x^s f(x) - \partial_x^s f(x_0)|, \;\; x \to x_0 \text{ in } U.$$

This definition says that the order of vanishing of derivatives of a function is only polynomially worse than that of lower derivatives. Every analytic function is in one of the 0-Gevrey classes $\mathcal{G}_\tau^0$ for some $\tau < \infty$, but many more functions are as well. For example, the function

$$f(x) = \begin{cases} \exp(-1/x^p), \text{ for } x > 0, \\ 0, \text{ for } x \leqslant 0 \end{cases}$$

is in $\mathcal{G}_{p+1}^0$, but

$$f(x) = \begin{cases} \exp(-\exp(1/x)), \text{ for } x > 0, \\ 0, \text{ for } x \leqslant 0 \end{cases}$$

is not in any 0-Gevrey class for finite τ. This implies that the 0-Gevrey class contains a rich subset of functions with compact support as well as functions which are constant on intervals.

The 0-Gevrey class assumption will only come in to play in the case of infinitely degenerate critical points (see Subsection 3.4).

2.3. **Semiclassical calculus with 2 parameters.** Following Sjöstrand-Zworski [SZ07, §3.3] and [CW11], we introduce a calculus with two parameters. We will not present the proofs in the following lemmas, as they have appeared in several other places, but merely include the statements for the reader's convenience, as well as pointers to where proofs can be found.

For $\alpha \in [0,1]$ and $\beta \leqslant 1-\alpha$, we let

$$\begin{aligned} &\mathcal{S}^{k,m,\widetilde{m}}_{\alpha,\beta}\left(T^*(\mathbb{R}^n)\right) := \\ &= \Bigg\{ a \in \mathcal{C}^\infty\left(\mathbb{R}^n \times (\mathbb{R}^n)^* \times (0,1]^2\right) : \\ &\qquad \left|\partial_x^\rho \partial_\xi^\gamma a(x,\xi;h,\tilde{h})\right| \leqslant C_{\rho\gamma} h^{-m}\tilde{h}^{-\widetilde{m}} \left(\frac{\tilde{h}}{h}\right)^{\alpha|\rho|+\beta|\gamma|} \langle \xi \rangle^{k-|\gamma|} \Bigg\}. \end{aligned}$$

Throughout this work we will always assume $\tilde{h} \geqslant h$. We let $\Psi^{k,m,\widetilde{m}}_{\alpha,\beta}$ denote the corresponding spaces of semiclassical pseudodifferential operators obtained by Weyl quantization of these symbols. We will sometimes add a subscript of h or $\tilde{h}$ to indicate which parameter is used in the quantization; in the absence of such a parameter, the quantization is assumed to be in h. The class $\mathcal{S}_{\alpha,\beta}$ (with no superscripts) will denote $\mathcal{S}^{0,0,0}_{\alpha,\beta}$ for brevity.

In [SZ07] (for the homogeneous case $\alpha = \beta = 1/2$), and in [CW11] (for the inhomogeneous case $\alpha \neq \beta$), it is observed that the composition in the calculus can be computed in terms of a symbol product that converges in the sense that terms improve in $\tilde{h}$ and ξ orders, but not in h orders. This happens because when $\alpha+\beta = 1$, the $(h^{-\alpha}, h^{-\beta})$ calculus is marginal, which is what the rescaling (blowup) and introduction of the second parameter $\tilde{h}$ accomplishes. In the sequel, we will always assume we are in the inhomogeneous marginal case:

$$\alpha + \beta = 1.$$

If $\alpha + \beta < 1$, then of course the calculus is no longer marginal and computations become much easier.

By the same arguments employed in [SZ07] (see [CW11]), we may easily verify that the calculus $\Psi_{\alpha,\beta}$ is closed under composition: if $a \in \mathcal{S}^{k,m,\widetilde{m}}_{\alpha,\beta}$ and $b \in \mathcal{S}^{k',m',\widetilde{m}''}_{\alpha,\beta}$ then

$$\operatorname{Op}_h^w(a) \circ \operatorname{Op}_h^w(b) = \operatorname{Op}_h^w(c) \ \text{ with } \ c \in \mathcal{S}^{k+k',m+m',\widetilde{m}+\widetilde{m}'}_{\alpha,\beta}.$$

In addition, as in [CW11], we have a symbolic expansion for c in powers of $\tilde{h}$.

We have the following Lemma from [CW11], which is a more general version of [SZ07, Lemma 3.6]:

Lemma 2.2. *Suppose that $a, b \in \mathcal{S}_{\alpha,\beta}$, and that $c^w = a^w \circ b^w$. Then*

$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k a(x,\xi)b(y,\eta)|_{x=y,\xi=\eta} + e_N(x,\xi)\,, \tag{2.1}$$

where for some M

$$
\begin{aligned}
&|\partial^\gamma e_N| \leqslant C_N h^{N+1} \\
&\quad \times \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n \\ (y,\eta)\in T^*\mathbb{R}^n}} \sup_{|\rho|\leqslant M\,,\rho\in\mathbb{N}^{4n}} \left|\Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^{N+1} a(x,\xi) b(y,\eta)\right|\,,
\end{aligned}
\tag{2.2}
$$

where $\sigma(D) = \sigma(D_x, D_\xi; D_y, D_\eta)$ *as usual, and*

$$
\Gamma_{\alpha,\beta,\rho,\gamma}(D) = (h^\alpha \partial_{(x,y)}, h^\beta \partial_{(\xi,\eta)}))^\rho \partial^{\gamma_1}\partial^{\gamma_2}.
$$

As a particular consequence we notice that if $a \in \mathcal{S}_{\alpha,\beta}(T^*\mathbb{R}^n)$ and $b \in \mathcal{S}(T^*\mathbb{R}^n)$ then

$$
\begin{aligned}
c(x,\xi) = &\sum_{k=0}^{N} \frac{1}{k!} \left(ih\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k a(x,\xi) b(y,\eta)|_{x=y,\xi=\eta} \\
&+ \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}\left(h^{N+1} \max\left\{(\tilde{h}/h)^{(N+1)\alpha}, (\tilde{h}/h)^{(N+1)\beta}\right\}\right)\,.
\end{aligned}
\tag{2.3}
$$

We will let $\mathcal{B}$ denote the "blowdown map"

$$
(x,\xi) = \mathcal{B}(X,\Xi) = ((h/\tilde{h})^\alpha X, (h/\tilde{h})^\beta \Xi).
\tag{2.4}
$$

The spaces of operators Ψ_h and $\Psi_{\tilde{h}}$ are related via a unitary rescaling in the following fashion. Let $a \in \mathcal{S}^{k,m,\tilde{m}}_{\alpha,\beta}$, and consider the rescaled symbol

$$
a\left(\left(h/\tilde{h}\right)^\alpha X, \left(h/\tilde{h}\right)^\beta \Xi\right) = a\circ\mathcal{B} \in \mathcal{S}^{k,m,\tilde{m}}_{0,0}.
$$

Define the unitary operator $T_{h,\tilde{h}}u(X) = \left(h/\tilde{h}\right)^{\frac{n\alpha}{2}} u\left(\left(h/\tilde{h}\right)^\alpha X\right)$, so that

$$
\mathrm{Op}\,^w_{\tilde{h}}(a\circ B) T_{h,\tilde{h}} u = T_{h,\tilde{h}} \mathrm{Op}\,^w_h(a) u.
$$

3. The trapping

In order to prove Theorem 2 and Corollary 2.1, we consider the critical points of the potential $V(x)$, or more specifically the critical points of the principal part, $V_0(x) = A^{-2}$, of $V = A^{-2} + h^2 V_1$. The assumption that the trapped set has only finitely many connected components implies that the potential $V_0(x)$ has only finitely many critical *values*. We break the analysis of the critical values into those for which the Hamiltonian flow of the principal part of our symbol, $p_0 = \xi^2 + V_0(x)$, is locally unstable (either "genuinely" unstable or of transmission inflection type), and those for which the Hamiltonian flow is stable. This leads to the dichotomy in Theorem 2 and Corollary 2.1. The idea is that, if there is a critical value for which the Hamiltonian flow is stable, then we can immediately construct very good quasimodes and reach the second conclusions in Theorem 2 and Corollary 2.1. This is relatively straightforward and written in Subsection 3.6.

On the other hand, if there is no stable trapping, then all trapping is unstable, consisting of disjoint critical sets, and even if two critical sets exist at the same potential energy level, they must be separated by an unstable maximum critical value at a *higher* potential energy level (otherwise there would be a minimum in between, and hence at least weakly stable trapping), so they do not see each other. That is to say, the weakly stable/unstable manifolds of the separating maximum form a separatrix in the reduced phase space. This allows us to glue together microlocal estimates near each critical set, and the resolvent estimate is then simply the worst

of these estimates. Hence it suffices to classify microlocal resolvent estimates in a neighbourhood of any of these unstable critical sets. This is accomplished in Subsections 3.1-3.5. In this sense, this section contains a catalogue of microlocal resolvent estimates.

It is important to note at this point that for unstable trapping of finite degeneracy, the relevant resolvent estimates are all $o(h^{-2})$, that is to say, the sub-potential $h^2 V_1$ is always of lower order. If the trapping is unstable but infinitely degenerate, we need to work harder to absorb the sub-potential. The 0-Gevrey assumption will be important here.

3.1. **Unstable nondegenerate trapping.** Unstable nondegenerate trapping occurs when the potential V_0 has a nondegenerate maximum. As mentioned previously, let us for the time being consider the operator $\widetilde{Q} = -h^2\partial_x + V_0(x) - z$, where $V_0(x) = A^{-2}(x)$. To say that $x = 0$ is a nondegenerate maximum means that $x = 0$ is a critical point of $V_0(x)$ satisfying $V_0'(0) = 0$, $V_0''(0) < 0$, and then the Hamiltonian flow of $\tilde{q} = \xi^2 + V_0(x)$ near $(0,0)$ is

$$\begin{cases} \dot{x} = 2\xi, \\ \dot{\xi} = -V_0'(x) \sim x, \end{cases}$$

so that the stable/unstable manifolds for the flow are transversal at the critical point $(0,0)$.

The following result as stated can be read off from [Chr07,Chr10,Chr11], and has also been studied in slightly different contexts in [CdVP94a,CdVP94b] and [BZ04], amongst many others. We only pause briefly to remark that, since the lower bound on the operator $\widetilde{Q}$ is of the order $h/\log(1/h) \gg h^2$, the same result applies equally well to $\widetilde{Q} + h^2 V_1$.

Proposition 3.1. *Suppose $x = 0$ is a nondegenerate local maximum of the potential V_0, $V_0(0) = 1$. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_\epsilon \frac{h}{\log(1/h)} \|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{3.1}$$

3.2. **Unstable finitely degenerate trapping.** In this subsection, we consider an isolated critical point leading to unstable but finitely degenerate trapping. That is, we now assume that $x = 0$ is a degenerate maximum for the function $V_0(x) = A^{-2}(x)$ of order $m \geqslant 2$. If we again assume $V_0(0) = 1$, then this means that near $x = 0$, $V_0(x) \sim 1 - x^{2m}$. Critical points of this form were studied in [CW11], but the proof can also be more or less deduced from the proofs of Propositions 3.6 and 3.8 below. We only remark briefly that again, since the lower bound on the operator $\widetilde{Q}$ is of the order $h^{2m/(m+1)} \gg h^2$, the estimate applies equally well to $\widetilde{Q} + h^2 V_1$.

Proposition 3.2. *Let $\widetilde{Q} = -h^2\partial_x^2 + V_0(x) - z$. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_\epsilon h^{2m/(m+1)} \|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{3.2}$$

Remark 3.3. In [CW11], it is also shown that this estimate is sharp in the sense that the exponent $2m/(m+1)$ cannot be improved.

3.3. **Finitely degenerate inflection transmission trapping.** We next study the case when the potential has an inflection point of finitely degenerate type. That is, let us assume the point $x = 1$ is a finitely degenerate inflection point, so that locally near $x = 1$, the potential $V_0(x) = A^{-2}(x)$ takes the form

$$V_0(x) \sim C_1^{-1} - c_2(x-1)^{2m_2+1}, \; m_2 \geqslant 1$$

where $C_1 > 1$ and $c_2 > 0$. Of course the constants are arbitrary (chosen to agree with those in [CM13]), and c_2 could be negative without changing much of the analysis. This Proposition and the proof are in [CM13], and as we will once again revisit the proof of this Proposition in Subsection 3.5, we will omit it at this point. But one last time, let us observe that since the lower bound on the operator $\widetilde{Q}$ is of the order $h^{(4m_2+2)/(2m_2+3)} \gg h^2$, the estimate applies equally well to the operator $\widetilde{Q} + h^2 V_1$.

Proposition 3.4. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x-1,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_\epsilon h^{(4m_2+2)/(2m_2+3)} \|\varphi^w u\|, \; z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon]. \tag{3.3}$$

Remark 3.5. We remark that in this case, [CM13] shows once again that this estimate is sharp in the sense that the exponent $(4m_2+2)/(2m_2+3)$ cannot be improved.

3.4. **Unstable infinitely degenerate and cylindrical trapping.** In this subsection, we study the case where the principal part of the potential $V(x) = A^{-2}(x) + h^2 V_1(x)$ has an infinitely degenerate maximum, say, at the point $x = 0$. Let $V_0(x) = A^{-2}(x)$. As usual, we again assume that $V_0(0) = 1$, so that

$$V_0(x) = 1 - \mathcal{O}(x^\infty)$$

in a neighbourhood of $x = 0$. Of course this is not very precise, as V_0 could be constant in a neighbourhood of $x = 0$ and still satisfy this, and the proof must be modified to suit these two cases. So let us first assume that $V_0(0) = 1$, and $V_0'(x)$ vanishes to infinite order at $x = 0$, however, $\pm V_0'(x) < 0$ for $\pm x > 0$. That is, the critical point at $x = 0$ is infinitely degenerate but isolated.

Our microlocal spectral theory result is then that the microlocal cutoff resolvent is bounded by $\mathcal{O}_\eta(h^{-2-\eta})$ for any $\eta > 0$. In order to state the result, let

$$\widetilde{Q} = -h^2\partial_x^2 + V(x) - z = -h^2\partial_x^2 + A^{-2}(x) + h^2 V_1(x) - z.$$

Proposition 3.6. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C_{\epsilon,\eta} > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_{\epsilon,\eta} h^{2+\eta} \|\varphi^w u\|, \; z \in [1-\epsilon, 1+\epsilon]. \tag{3.4}$$

Remark 3.7. As this is the limiting case as $m \to \infty$ of Proposition 3.2, we believe the optimal lower bound in this case is $h^2/\gamma(h)$ for some $\gamma(h) \to 0$. This is further suggested by a microlocal scaling heuristic. However, various attempts to tighten up the argument to get the better lower bound seem to fail. It would be very interesting to determine if a lower bound of $h^2/\gamma(h)$ or even h^2 holds.

For our next result, we consider the case where there is a whole cylinder of unstable trapping. That is, we assume the principal part of the effective potential $V_0(x)$ has a maximum $V_0(x) \equiv 1$ on an interval, say $x \in [-a, a]$, and that $\pm V_0'(x) <$

0 for $\pm x > a$. Our main result in this case says that the microlocal cutoff resolvent is again controlled by $h^{-2-\eta}$ for any $\eta > 0$. Let us again set

$$\widetilde{Q} = -h^2 \partial_x^2 + V(x) - z.$$

Proposition 3.8. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|x| \leqslant a + \epsilon, |\xi| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C_{\epsilon,\eta} > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_{\epsilon,\eta} h^{2+\eta} \|\varphi^w u\|, \;\; z \in [1-\epsilon, 1+\epsilon]. \tag{3.5}$$

Remark 3.9. For similar reasons, we expect the optimal lower bound in this case should be h^2.

Proof. The proof of these Propositions is very similar, so we put them together. We will first prove Proposition 3.6, and then point out how the proof must be modified to get Proposition 3.8.

The idea of the proof of Proposition 3.6 (and indeed Proposition 3.8) is to add a small h-dependent bump with a *finitely degenerate* maximum, and then use the result of Proposition 3.2. Of course the bump has to be sufficiently small that the operator $\widetilde{Q}$ is close to the perturbed operator.

Choose a point $x_0 = x_0(h) > 0$ and $\epsilon > 0$ so that x_0 is the smallest point such that

$$-xV_0'(x) \geqslant \frac{h}{\varpi(h)}, \;\; x_0 \leqslant |x| \leqslant \epsilon,$$

where $\varpi(h)$ will be determined later. As long as $\varpi(h) \gg h$, this implies that $x_0 = o(1)$. We remark that, of course, x_0 depends also on the choice of $\varpi(h)$, but for any ϖ, there is such a choice, since V_0' vanishes to infinite order at $x = 0$. Further, as $V_0'(x) = \mathcal{O}(x^\infty)$ near $x = 0$, we have $x_0 \gg h^\delta$ for any $\delta > 0$. Fix $m \geqslant 2$ to be determined later in the proof (m will be large), and choose also an even function $f \in \mathcal{C}_c^\infty([-2,2]) \cap \mathcal{G}_\tau^0$ for some $\tau < \infty$, with $f(x) = 1 - \frac{1}{2m}x^{2m}$ for $|x| \leqslant 1$, and $f'(x) \leqslant 0$ or $x \in [0,2]$. For another parameter $\Gamma(h) > 0$ to be determined, let

$$W_h(x) = \Gamma(h) f(x/x_0),$$

and let

$$V_{0,h}(x) = V_0(x) + W_h(x)$$

and

$$V_h(x) = V(x) + W_h(x)$$

(see Figure 1). The parameter $\Gamma(h)$ will be seen to be $h^{2+\eta}$ for $\eta > 0$, $\eta = \mathcal{O}(m^{-1})$ as $m \to \infty$. By construction,

$$|V(x) - V_h(x)| \leqslant |W_h| \leqslant \Gamma(h).$$

Let $Q_1 = (hD)^2 + V_h$ with symbol $q_1 = \xi^2 + V_h$. The Hamilton vector field H associated to the symbol q_1 is given by

$$\begin{aligned} \mathsf{H} &= 2\xi\partial_x - V_h'\partial_\xi \\ &= 2\xi\partial_x - \left(\frac{\Gamma(h)}{x_0} f'(x/x_0) + V_0'(x) + h^2 V_1'(x) \right) \partial_\xi. \end{aligned}$$

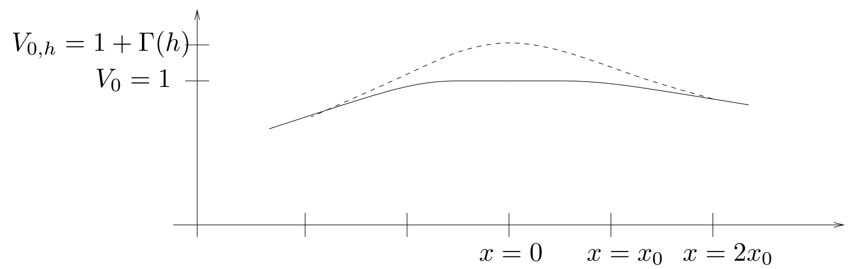

FIGURE 1. The potential V_0 and the modified potential $V_{0,h}$ (in dashed).

We will use the same change of coordinates and the same singular commutant as in [CW11], but we also have to track the loss coming from the coefficient $\Gamma(h)$. For $\alpha = 1/(m+1)$, let

$$\Xi = \frac{\xi}{(h/\tilde{h})^{m\alpha}}, \quad X = \frac{x}{(h/\tilde{h})^{\alpha}},$$

so that in the new blown-up coordinates Ξ, X,

$$\mathsf{H} = (h/\tilde{h})^{\frac{m-1}{m+1}} \big(\Xi\partial_X - (h/\tilde{h})^{(1-2m)/(m+1)} V_h'((h/\tilde{h})^{\alpha} X)\partial_\Xi\big) \tag{3.6}$$

Let $\Lambda(s)$ be defined as in [CW11] by fixing $\epsilon_0 > 0$ and setting

$$\Lambda(s) = \int_0^s \langle z \rangle^{-1-\epsilon_0}\, dz,$$

so that Λ is a zero order symbol satisfying $\Lambda(s) \sim s$ for s near 0. Following [Chr07, Chr11, CW11], we define

$$a(x,\xi;h) = \Lambda(\Xi)\Lambda(X)\chi(x)\chi(\xi) = \Lambda(\xi/(h/\tilde{h})^{m\alpha})\Lambda(x/(h/\tilde{h})^{\alpha})\chi(x)\chi(\xi),$$

where $\chi(s)$ is a cutoff function equal to 1 for $|s| < \delta_1$ and 0 for $s > 2\delta_1$ (δ_1 will be chosen shortly). Then a is bounded, and a 0 symbol in X, Ξ :

$$\left|\partial_X^{\alpha}\partial_\Xi^{\beta} a\right| \leqslant C_{\alpha,\beta}.$$

(Recall that $x = (h/\tilde{h})^{\alpha} X$ and $\xi = (h/\tilde{h})^{m\alpha}\Xi$.) Using (3.6), it is simple to compute

$$\begin{aligned}
\mathsf{H}(a) =& (h/\tilde{h})^{\frac{m-1}{m+1}} \chi(x)\chi(\xi)\big(\Lambda(\Xi)\langle X\rangle^{-1-\epsilon_0}\Xi \\
&- (h/\tilde{h})^{(1-2m)/(m+1)}\Lambda(X)V_h'((h/\tilde{h})^{\alpha}X)\langle\Xi\rangle^{-1-\epsilon_0}\big) + r \\
\equiv& (h/\tilde{h})^{\frac{m-1}{m+1}} g + r
\end{aligned} \tag{3.7}$$

with

$$\operatorname{supp} r \subset \{|x| > \delta_1\} \cup \{|\xi| > \delta_1\}$$

(r comes from terms involving derivatives of $\chi(x)\chi(\xi)$).

For $|X| \leqslant (h/\tilde{h})^{-\alpha} x_0$, we have

$$\begin{aligned}
-(h/\tilde{h})^{(1-2m)/(m+1)} &\Lambda(X) V_h'((h/\tilde{h})^{\alpha}X)\langle\Xi\rangle^{-1-\epsilon_0} \\
&= \Gamma(h) x_0^{-2m}\Lambda(X) X^{2m-1}\langle\Xi\rangle^{-1-\epsilon_0} + g_2,
\end{aligned}$$

with

$$g_2 = -(h/\tilde{h})^{(1-2m)/(m+1)}\Lambda(X)(V_0' + h^2 V_1')((h/\tilde{h})^{\alpha}X)\langle\Xi\rangle^{-1-\epsilon_0}.$$

Note that we always have $-\Lambda(x)V_0'(x) \geqslant 0$, so we expect the quantization of g_2 to be at least semibounded below. This is demonstrated in Lemma 3.11 below.

For $|X| \leqslant (h/\tilde{h})^{-\alpha}x_0$ and $|\Xi| \leqslant (h/\tilde{h})^{-\alpha m}\delta_1$ consider

$$\begin{aligned} g =& \chi(x)\chi(\xi)\big(\Lambda(\Xi)\langle X\rangle^{-1-\epsilon_0}\Xi \\ & - (h/\tilde{h})^{(1-2m)/(m+1)}\Lambda(X)V_h'((h/\tilde{h})^{\alpha}X)\langle\Xi\rangle^{-1-\epsilon_0} \\ =& \Lambda(\Xi)\Xi\langle X\rangle^{-1-\epsilon_0} + \Gamma(h)x_0^{-2m}\Lambda(X)X^{2m-1}\langle\Xi\rangle^{-1-\epsilon_0} + g_2 \\ =& \lambda^2\left(\lambda^{-1}\Lambda(\Xi)(\lambda^{-1}\Xi)\langle X\rangle^{-1-\epsilon_0} + \lambda^{-2m-2}\Gamma x_0^{-2m}\lambda\Lambda(X)(\lambda X)^{2m-1}\langle\Xi\rangle^{-1-\epsilon_0}\right) + g_2 \\ =& \lambda^2\left(\widetilde{\Lambda}_1(\Xi')\Xi'\left\langle\lambda^{-1}X'\right\rangle^{-1-\epsilon_0} + \lambda^{-2m-2}\Gamma x_0^{-2m}\widetilde{\Lambda}_2(X')(X')^{2m-1}\langle\lambda\Xi'\rangle^{-1-\epsilon_0}\right) + g_2 \\ =:& g_1 + g_2, \end{aligned}$$

where we have used the L^2-unitary rescaling

$$X' = \lambda X, \ \ \Xi' = \lambda^{-1}\Xi,$$

and $\lambda > 0$ (small) will be determined in the course of the proof.

The functions $\widetilde{\Lambda}_j$, $j = 1, 2$, are defined by changing variables:

$$\widetilde{\Lambda}_1(\Xi') = \lambda^{-1}\Lambda(\Xi) = \lambda^{-1}\Lambda(\lambda\Xi'),$$

and

$$\widetilde{\Lambda}_2(X') = \lambda\Lambda(X) = \lambda\Lambda(\lambda^{-1}X').$$

The error term g_2 is the term in the expansion of g coming from estimating using W_h' rather than V_h'. We will deal with g_2 in due course. We are now microlocalized on a set where

$$|X'| \leqslant \lambda(h/\tilde{h})^{-\alpha}x_0, \ \ |\Xi'| \leqslant \lambda^{-1}(h/\tilde{h})^{-m\alpha}\delta_1,$$

and will be quantizing in the $\tilde{h}$-Weyl calculus, so we need symbolic estimates on these sets.

If

$$|\lambda^{-1}X'| \leqslant \delta_1, \ \text{and } |\lambda\Xi'| \leqslant \delta_1,$$

and $\delta_1 > 0$ is sufficiently small, then $\widetilde{\Lambda}_1(\Xi') \sim \Xi'$ and $\widetilde{\Lambda}_2(X') \sim X'$, so that g_1 is bounded below as follows:

$$g_1 \geqslant \min\left\{\lambda^2, \lambda^{-2m}\Gamma x_0^{-2m}\right\}((\Xi')^2 + (X')^{2m}). \tag{3.8}$$

Then the $\tilde{h}$-quantization of g_1 is bounded below microlocally on this set by this minimum times $\tilde{h}^{2m/(m+1)}$ (see [CW11, Lemma A.2]).

Now on the complementary set, we have one of either $|\lambda^{-1}X'|^{1+\epsilon_0}$ or $|\lambda\Xi'|$ is larger than, say, $(\delta_1/2)^{1+\epsilon_0}$. We also need to keep track of the *relative* size of these two quantities. If $|\lambda\Xi'| \geqslant \max\left(\left|\lambda^{-1}X'\right|^{1+\epsilon_0}, (\delta_1/2)^{1+\epsilon_0}\right)$ then

$$\begin{aligned} g_1 &\geqslant \lambda^2\widetilde{\Lambda}_1(\Xi')\Xi'\left\langle\lambda^{-1}X'\right\rangle^{-1-\epsilon_0} \\ &\geqslant c\lambda^2\widetilde{\Lambda}_1(\Xi')\frac{\Xi'}{|\lambda\Xi'|} \\ &= c\lambda\widetilde{\Lambda}_1(\Xi')\operatorname{sgn}(\Xi') \\ &= c\Lambda(\lambda\Xi')\operatorname{sgn}(\Xi') \\ &\geqslant c_{\delta_1}. \end{aligned} \tag{3.9}$$

Hence the $\tilde{h}$-quantization of g_1 is bounded below by a positive constant, independent of h and $\tilde{h}$ on this set.

The remaining set is a bit more difficult. If

$$\left|\lambda^{-1}X'\right|^{1+\epsilon_0} \geqslant \max\left(|\lambda\Xi'|, (\delta_1/2)^{1+\epsilon_0}\right),$$

then

$$\begin{aligned}
g_1 &\geqslant \lambda^2\Big(\widetilde{\Lambda}_1(\Xi')\Xi'\left\langle (h/\tilde{h})^{-\alpha}x_0\right\rangle^{-1-\epsilon_0} \\
&\quad + \lambda^{-2m-2}\Gamma x_0^{-2m}\widetilde{\Lambda}_2(X')(X')^{2m-1}\left\langle\lambda\Xi'\right\rangle^{-(1+\epsilon_0)}\Big) \\
&= \lambda^2\Big(\lambda^{-1}\Lambda(\lambda\Xi')\Xi'\left\langle (h/\tilde{h})^{-\alpha}x_0\right\rangle^{-1-\epsilon_0} \\
&\quad + \lambda^{-2m-2}\Gamma x_0^{-2m}\lambda\Lambda(\lambda^{-1}X')(X')^{2m-1}\left\langle\lambda\Xi'\right\rangle^{-(1+\epsilon_0)}\Big) \\
&= \lambda^2\Big(\lambda^{-1}\Lambda(\lambda\Xi')\Xi'\left\langle (h/\tilde{h})^{-\alpha}x_0\right\rangle^{-1-\epsilon_0} \\
&\quad + \lambda^{-2m}\Gamma x_0^{-2m}\Lambda(\lambda^{-1}X')(X')^{2m-2}(\lambda^{-1}X')\left\langle\lambda\Xi'\right\rangle^{-(1+\epsilon_0)}\Big) \\
&\geqslant \begin{cases} c_{\delta_1}\min\{\lambda^2(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0}, \lambda^{-2m+2}\Gamma x_0^{-2m}\} \\ \quad\times((\Xi')^2+(X')^{2m-2}),\ \text{if } |\lambda\Xi'| \leqslant (\delta_1/2)^{1+\epsilon_0}, \\ c'_{\delta_1}(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0},\ \text{if } |\lambda\Xi'| \geqslant (\delta_1/2)^{1+\epsilon_0}. \end{cases}
\end{aligned} \tag{3.10}$$

We now optimize the minimum in (3.10) to determine λ in terms of the other parameters:

$$\lambda^2(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0} = \lambda^{-2m+2}\Gamma x_0^{-2m},$$

or

$$\lambda^2 = \Gamma^{1/m}(h/\tilde{h})^{-\alpha(1+\epsilon_0)/m}x_0^{-2+(1+\epsilon_0)/m}.$$

Then the minimum is

$$\lambda^2(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0} = \Gamma^{1/m}(h/\tilde{h})^{\alpha(1+\epsilon_0)(m-1)/m}x_0^{-3-\epsilon_0+(1+\epsilon_0)/m},$$

and the $\tilde{h}$-quantization of g_1 on this set is bounded below microlocally by this number times $\tilde{h}^{2(m-1)/m}$ (see [CW11, Lemma A.2]).

Remark 3.10. We pause to remark that here is one place where alternative methods to optimize the lower bounds give worse results. For example, on the set where

$$|\lambda\Xi'| \leqslant (\delta_1/2)^{1+\epsilon_0} \leqslant |\lambda^{-1}X'|^{1+\epsilon_0},$$

we could estimate g_1 from below using only the second term. This gives a lower bound of $c''_{\delta_1}\Gamma x_0^{-2m}$, which is much worse than that computed above.

Finally, recalling that eventually $\tilde{h} > 0$ will be fixed and $h \ll \tilde{h}$, taking the worst lower bound from (3.8) through (3.10), we obtain for a function u with h-wavefront set contained in the set where $|\lambda^{-1}X'| \leqslant (h/\tilde{h})^{-\alpha}x_0$, $|\lambda\Xi'| \leqslant (h/\tilde{h})^{-m\alpha}\delta_1$,

$$\left\langle \operatorname{Op}_{\tilde{h}}(g_1)u, u\right\rangle \geqslant \Gamma^{1/m}(h/\tilde{h})^{\alpha(1+\epsilon_0)(m-1)/m}x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}\tilde{h}^{2(m-1)/m}\|u\|^2.$$

On the other hand, if $(h/\tilde{h})^{-\alpha}x_0 \leqslant |X| \leqslant (h/\tilde{h})^{-\alpha}\delta_1$, we have

$$\begin{aligned}
&-(h/\tilde{h})^{(1-2m)/(m+1)}\Lambda(X)V_h'((h/\tilde{h})^\alpha X)\langle\Xi\rangle^{-1-\epsilon_0} \\
&\quad = -(h/\tilde{h})^{(1-2m)/(m+1)}\operatorname{sgn}(X)B(X)\frac{|(h/\tilde{h})^\alpha X|}{|(h/\tilde{h})^\alpha X|}V'((h/\tilde{h})^\alpha X)\langle\Xi\rangle^{-1-\epsilon_0} + g_3 \\
&\quad \geqslant (h/\tilde{h})^{(1-2m)/(m+1)}\left(\frac{B(X)}{|(h/\tilde{h})^\alpha X|}\frac{h}{\varpi(h)} - \mathcal{O}(h^2)\right)\langle\Xi\rangle^{-1-\epsilon_0} + g_3 \\
&\quad \geqslant C\frac{h^{(2-m)/(m+1)}\tilde{h}^{(2m-1)/(m+1)}}{\varpi(h)}\langle\Xi\rangle^{-1-\epsilon_0} + g_3 \\
&\quad \geqslant C\frac{h^{(2-m)/(m+1)}\tilde{h}^{(2m-1)/(m+1)}}{\varpi(h)}(h/\tilde{h})^{(1+\epsilon_0)m/(1+m)} + g_3 \\
&\quad \geqslant 2\frac{h^{3/(m+1)}\tilde{h}^{(m-1)/(m+1)}}{\varpi(h)} + g_3, \qquad (3.11)
\end{aligned}$$

where $B(X) \geqslant c_0 > 0$. The second inequality holds provided $h/\varpi \gg h^2$ (so that h^2V_1' is controlled by V_0'), and the last inequality holds as $h \to 0$ provided $\epsilon_0 < 1/m$. The error $g_3 \geqslant 0$ comes from using V' in the expansion of g rather than W_h'.

We now deal with the (nearly) positive error terms g_2 and g_3.

Lemma 3.11. *The error terms g_2 and g_3 are semi-bounded below in the following sense: if $u(X)$ has wavefront set localized in*

$$\{|X| \leqslant \epsilon(h/\tilde{h})^{-1/(m+1)},\ |\Xi| \leqslant \epsilon(h/\tilde{h})^{-m/(m+1)}\},$$

then for any $\delta > 0$ and $N > 0$,

$$\langle \operatorname{Op}_{\tilde{h}}(g_j)u, u\rangle \geqslant -C_N h^{(N-2)m/(m+1)-\delta}\tilde{h}^{2m/(m+1)}\|u\|^2,$$

for $j = 2, 3$.

Proof. We prove the relevant bounds for $x \geqslant 0$. The analysis for $x \leqslant 0$ is similar. For g_2, for $N > 0$ large, and $\delta > 0$ small, choose $0 < x_1 < x_2 = o(1)$ satisfying

$$-x_1V_0'(x_1) = h^{Nm/(m+1)}$$

and

$$-x_2V_0'(x_2) = h^{Nm/(m+1)-\delta}.$$

As usual, since $V_0'(x) = \mathcal{O}(x^\infty)$, the points x_j, $j = 1, 2$ satisfy $x_j \gg h^{\delta_2}$ for any $\delta_2 > 0$. The 0-Gevrey condition also implies $|x_2 - x_1| \gg h^{\delta_2}$ for any $\delta_2 > 0$ as well. To see this, Taylor's theorem says

$$(V_0'(x_2) - V_0'(x_1)) = V_0''(\xi)(x_2 - x_1)$$

for some $x_1 \leqslant \xi \leqslant x_2$. The 0-Gevrey condition and monotonicity near $x = 0$ implies

$$|V_0''(\xi)| \leqslant |V_0''(x_2)| \leqslant C\left|\frac{V_0'(x_2)}{x_2^\tau}\right|,$$

so that

$$(V_0'(x_2) - V_0'(x_1)) \leqslant C\left|\frac{V_0'(x_2)}{x_2^\tau}\right|(x_2 - x_1)$$

for some $\tau < \infty$, which in turn implies (recalling $V_0' < 0$ for $x > 0$ near 0)

$$(x_2 - x_1) \geqslant C'\frac{V_0'(x_1) - V_0'(x_2)}{|V_0'(x_2)|}x_2^\tau = x_2^\tau\left(1 - \left|\frac{V_0'(x_1)}{V_0'(x_2)}\right|\right).$$

We claim

$$\left| \frac{V_0'(x_1)}{V_0'(x_2)} \right| = o(1),$$

which will finish the proof that $|x_2 - x_1| \gg h^{\delta_2}$ for any $\delta_2 > 0$. For this, we write

$$\begin{aligned} \left| \frac{V_0'(x_1)}{V_0'(x_2)} \right| &= \left| x_1 \frac{V_0'(x_1)}{x_2 V_0'(x_2)} \right| \frac{x_2}{x_1} \\ &= h^\delta \frac{x_2}{x_1}. \end{aligned}$$

Writing $x_2 = x_1 + \gamma(h)$, we are trying to show $\gamma(h) \gg h^{\delta_2}$ for any $\delta_2 > 0$. For a fixed δ_2, if $\gamma(h) \gg h^{\delta_2}$ we're done. If $\gamma(h) < h^{\delta_2}$, then we will produce a contradiction (in fact showing that $\gamma(h) \gg h^{\delta_2}$). If $\gamma(h) < h^{\delta_2}$ for this δ_2, then

$$\frac{\gamma(h)}{x_1} \ll 1,$$

since $x_1 \gg h^{\delta_2}$. Then it follows that

$$h^\delta \frac{x_2}{x_1} = h^\delta \frac{x_1 + \gamma(h)}{x_1} \ll 2h^\delta = o(1).$$

Plugging into our earlier computation, we get

$$x_2 - x_1 = C' x_2^\tau (1 - o(1)) \gg C'_{\delta_3} h^{\tau \delta_3}$$

for any $\delta_3 > 0$. Taking $\delta_3 > 0$ sufficiently small so that $h^{\tau\delta_3} \gg h^{\delta_2}$ implies

$$\gamma(h) = x_2 - x_1 \gg h^{\delta_2},$$

which is a contradiction to our assumption that $\gamma(h) \leqslant h^{\delta_2}$.

Now let $\psi(x)$ be a smooth function, $\psi \geqslant 0$, $\psi(x) \equiv 1$ on $[0, x_1]$ with $\psi(x) \equiv 0$ for $x \geqslant x_2$. Assume also that $|\partial_x^k \psi| \leqslant C_k |x_2 - x_1|^{-k} = o(h^{-k\delta_2})$ for any $\delta_2 > 0$. Let $\tilde{\psi}(X) = \psi((h/\tilde{h})^\alpha X)$, $\alpha = 1/(m+1)$, so that

$$|\partial_X^k \tilde{\psi}| \leqslant C_k (h/\tilde{h})^{\alpha k} \cdot o(h^{-k\delta_2}).$$

We have

$$\begin{aligned} &\left\langle \mathrm{Op}_{\tilde{h}}(g_2) u, u \right\rangle \\ &\quad = \left\langle \mathrm{Op}_{\tilde{h}}(g_2)(1 - \tilde{\psi})u, (1 - \tilde{\psi})u \right\rangle + \left\langle \mathrm{Op}_{\tilde{h}}(g_2) \tilde{\psi} u, \tilde{\psi} u \right\rangle \\ &\quad\quad + 2 \left\langle \mathrm{Op}_{\tilde{h}}(g_2) \tilde{\psi} u, (1 - \tilde{\psi})u \right\rangle. \end{aligned}$$

We estimate each term separately.

On the support of $1 - \tilde{\psi}$ (again recalling we are only looking at $x \geqslant 0$), we have $(h/\tilde{h})^{1/(m+1)} X \geqslant x_1$ so that in this region we can apply the 0-Gevrey condition to V_1' to absorb $h^2 V_1'$ into V_0'. Recall that V_1 consists of quotients of derivatives of A with powers of A. The function A is bounded above and below by a (positive) constant for x small, so we are really only concerned with estimating a finite number of derivatives of A. Then according to the 0-Gevrey condition, for any $\delta_2 > 0$, we have for some $s, \tau < \infty$

$$\begin{aligned} h^2 |V_1'((h/\tilde{h})^\alpha X)| &\leqslant C h^2 |x_1|^{-s\tau} |A'((h/\tilde{h})^\alpha X)| \\ &\leqslant C h^{2 - s\tau\delta_2} |V_0'((h/\tilde{h})^\alpha X)|, \end{aligned}$$

and similarly for a finite number of derivatives of V_1. By taking $\delta_2 > 0$ sufficiently small we see that on the support of $1-\tilde{\psi}$, the quantization of V_0' controls that of $h^2 V_1'$. That is, for $h > 0$ sufficiently small,

$$\begin{aligned}
&\left\langle \mathrm{Op}_{\tilde{h}}(g_2)(1-\tilde{\psi})u, (1-\tilde{\psi})u \right\rangle \\
&\quad \geqslant -C\left(\frac{h}{\tilde{h}}\right)^{\frac{(1-2m)}{(m+1)}} \left\langle \mathrm{Op}_{\tilde{h}}(\Lambda(X)V_0'((h/\tilde{h})^{\alpha}X)\langle \Xi \rangle^{-1-\epsilon_0})(1-\tilde{\psi})u, (1-\tilde{\psi})u \right\rangle .
\end{aligned}$$

Then we calculate in this region

$$\begin{aligned}
&\left(\frac{h}{\tilde{h}}\right)^{(1-2m)/(m+1)} \left(\frac{\Lambda(X)}{(h/\tilde{h})^{1/(m+1)}X}\right) \\
&\quad \times \left(-(h/\tilde{h})^{1/(m+1)}XV_0'((h/\tilde{h})^{1/(m+1)}X)\right)\langle \Xi \rangle^{-1-\epsilon_0} \\
&\quad = h^{(1-2m)/(m+1)}\tilde{h}^{(2m-1)/(m+1)}h^{Nm/(m+1)}A(X,h,\tilde{h})\langle \Xi \rangle^{-1-\epsilon_0}
\end{aligned}$$

where A is a symbol bounded below by a positive constant. This follows since

$$\begin{aligned}
X &\geqslant \left(\frac{h}{\tilde{h}}\right)^{-\alpha} x_1 \\
&\geqslant \left(\frac{h}{\tilde{h}}\right)^{-\alpha} h^{\delta_2}
\end{aligned}$$

for any $\delta_2 > 0$. Taking $\delta_2 < \alpha$, this lower bound is (at least) a positive constant.

On the set where $A\langle \Xi \rangle^{-1-\epsilon_0} \geqslant 1$, this operator is bounded below, while on the complement, we use the Sharp Gårding inequality to get for any $\delta_2 > 0$

$$\begin{aligned}
&\left\langle \mathrm{Op}_{\tilde{h}}(g_2)(1-\tilde{\psi})u, (1-\tilde{\psi})u \right\rangle \\
&\quad \geqslant -C_{\delta_2}\tilde{h}h^{((N-2)m+2)/(m+1)-2\delta_2}\tilde{h}^{(2m-2)/(m+1)}\|(1-\tilde{\psi})u\|^2.
\end{aligned}$$

For the remaining two terms, on the support of $\tilde{\psi}$, we have $0 \leqslant (h/\tilde{h})^{1/(m+1)}X \leqslant x_2$. We know that $|\partial_x^k A|$ is an increasing function for small x, so that to estimate V_1', we estimate a finite number of derivatives of A from above, we can estimate at the right-hand endpoint x_2. That is, we have as above for $s, \tau < \infty$ and any $\delta_2 > 0$,

$$\begin{aligned}
h^2|V_1'((h/\tilde{h})^{\alpha}X)| &\leqslant Ch^2|x_2|^{-s\tau}|V_0'(x_2)| \\
&\leqslant Ch^2|x_2|^{-s\tau-1}|x_2V_0'(x_2)| \\
&\leqslant Ch^{2-\delta_2(1+s\tau)}h^{Nm/(m+1)-\delta}
\end{aligned}$$

by our choice of x_2. This implies that on the support of $\tilde{\psi}$, h^2V_1' is controlled by a large power of h, by taking $\delta_2 > 0$ sufficiently small. That is, in this region

$$\begin{aligned}
g_2 = (h/\tilde{h})^{(1-2m)/(m+1)} &\Bigg[\left(\frac{\Lambda(X)}{(h/\tilde{h})^{\alpha)}X}\right)\left(-(h/\tilde{h})^{\alpha}XV_0'((h/\tilde{h})^{\alpha}X)\right) \\
&- h^2\Lambda(X)V_1'((h/\tilde{h})^{\alpha}X)\Bigg]\langle \Xi \rangle^{-1-\epsilon_0} \\
= h^{-2m/(m+1)}&\tilde{h}^{(2m)/(m+1)}h^{Nm/(m+1)-\delta}A_1(X,h,\tilde{h})\langle \Xi \rangle^{-1-\epsilon_0},
\end{aligned}$$

where A_1 is a function satisfying

$$|\partial_X^k A_1| \leqslant C_{k,\delta_2}(h^{1/(m+1)-\delta_2}\tilde{h}^{-1/(m+1)})^k.$$

Hence if $\delta_2 < 1/(m+1)$,

$$\left\langle \mathrm{Op}_{\tilde{h}}(g_2)\tilde{\psi}u, \tilde{\psi}u \right\rangle = \mathcal{O}(h^{(N-2)m/(m+1)-\delta}\tilde{h}^{2m/(m+1)})\|u\|^2,$$

and similarly

$$\left\langle \mathrm{Op}_{\tilde{h}}(g_2)\tilde{\psi}u, (1-\tilde{\psi})u \right\rangle = \mathcal{O}(h^{(N-2)m/(m+1)-\delta}\tilde{h}^{2m/(m+1)})\|u\|^2.$$

The proof for g_3 is the same (since we have assumed $f \in \mathcal{C}_c^\infty \cap \mathcal{G}_\tau^0$ for some $\tau < \infty$), but slightly easier, since g_3 is the error term coming from W_h away from $x = 0$, and W_h is already $\mathcal{O}(h^2)$.

□

Let us recap what we have shown so far and fix some of the parameters. We have perturbed our potential by a term of size Γ, which we want to be much smaller than our lower bound on $h\mathrm{Op}_h(\mathsf{H}(a))$. That is, we want to solve

$$h\left(\frac{h}{\tilde{h}}\right)^{(m-1)/(m+1)} \Gamma^{1/m}(h/\tilde{h})^{\alpha(1+\epsilon_0)(m-1)/m} x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}\tilde{h}^{2(m-1)/m} \gg \Gamma.$$

As m will be large, $\epsilon_0 < 1/m$, and $x_0 = o(1)$, it suffices to solve

$$\begin{aligned} h^{\frac{2m}{m+1}+\frac{(m-1)(1+\epsilon_0)}{m(m+1)}}\tilde{h}^{-\frac{(m-1)}{(m+1)}+2\frac{(m-1)}{m}-\frac{(m-1)(1+\epsilon_0)}{m(m+1)}} \\ = \Gamma^{(m-1)/m}, \end{aligned}$$

or

$$\Gamma = h^{2m^2/(m^2-1)+(1+\epsilon_0)/(m+1)}\tilde{h}^{2-m(m-1)/(m^2-1)-(1+\epsilon_0)/(m+1)}.$$

This means that for this value of Γ, our lower bound on $h\mathrm{Op}_h(\mathsf{H}(a))$ is

$$\begin{aligned} &\Gamma x_0^{-3-\epsilon_0+(1+\epsilon_0)/m} \\ &\quad = h^{2m^2/(m^2-1)+(1+\epsilon_0)/(m+1)}\tilde{h}^{2-m(m-1)/(m^2-1)-(1+\epsilon_0)/(m+1)}x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}. \end{aligned}$$

Observe that the exponent of h is $2 + \mathcal{O}(m^{-1})$ which can be made smaller than $2 + \eta$ for any $\eta > 0$ by taking m large.

We also have to choose the parameter $\varpi(h)$. For that we again match lower bounds:

$$\begin{aligned} \frac{h^{3/(m+1)}\tilde{h}^{(m-1)/(m+1)}}{\varpi(h)} &= \Gamma^{1/m}(h/\tilde{h})^{\alpha(1+\epsilon_0)(m-1)/m}x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}\tilde{h}^{(2m-2)/m} \\ &= h^{2m/(m^2-1)+(1+\epsilon_0)/(m+1)} \\ &\quad \times \tilde{h}^{4-2/m-(m-1)/(m^2-1)-(1+\epsilon_0)/(m+1)}x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}, \end{aligned}$$

or

$$\begin{aligned} \varpi(h) &= h^{3/(m+1)-2m/(m^2-1)-(1+\epsilon_0)/(m+1)} \\ &\quad \times \tilde{h}^{(m-1)/(m+1)-4+2/m+(m-1)/(m^2-1)+(1+\epsilon_0)/(m+1)}x_0^{3+\epsilon_0-(1+\epsilon_0)/m}. \end{aligned}$$

Taking m sufficiently large yields $\varpi(h)$ satisfying $h/\varpi(h) = o(1)$, so $\varpi(h) \gg h$, as required to determine $x_0(h)$ and fix all the parameters.

All told, we have shown for a function $u(X)$ with semiclassical wavefront set localized in a set $\{|X| \leqslant \epsilon(h/\tilde{h})^{-1/(m+1)}, |\Xi| \leqslant \epsilon(h/\tilde{h})^{-m/(m+1)}\}$

$$\begin{aligned}
& h(h/\tilde{h})^{(m-1)/(m+1)} \left\langle \mathrm{Op}_{\tilde{h}}(g)u, u \right\rangle \\
& \quad \geqslant C\Gamma(h) x_0^{-3-\epsilon_0+(1+\epsilon_0)/m} \|u\|^2 \\
& \qquad + h(h/\tilde{h})^{(m-1)/(m+1)} (\left\langle \mathrm{Op}_{\tilde{h}}(g_2)u, u \right\rangle + \left\langle \mathrm{Op}_{\tilde{h}}(g_3)u, u \right\rangle) \\
& \quad \geqslant C h^{2m^2/(m^2-1)+(1+\epsilon_0)/(m+1)} \tilde{h}^{2-m(m-1)/(m^2-1)-(1+\epsilon_0)/(m+1)} \\
& \qquad \times x_0^{-3-\epsilon_0+(1+\epsilon_0)/m} \|u\|^2 \\
& \qquad - C'_{N,\delta} \tilde{h}^{2m/(m+1)} h^{(N-2)m/(m+1)-\delta} h(h/\tilde{h})^{(m-1)/(m+1)} \|u\|^2 \\
& \quad \geqslant C'' h^{2m^2/(m^2-1)+(1+\epsilon_0)/(m+1)} \tilde{h}^{2-m(m-1)/(m^2-1)-(1+\epsilon_0)/(m+1)} \\
& \qquad \times x_0^{-3-\epsilon_0+(1+\epsilon_0)/m} \|u\|^2 .
\end{aligned}$$

We note that with these parameter values, of course $h/\varpi = o(1)$, which implies in turn that $x_0 = o(1)$, and since $h/\varpi \gg h^2$, the estimate (3.11) holds, which closes the argument.

This concludes the study of the principal term in the commutator expansion. Of course we still have to control the lower order terms in the commutator expansion, which we do in the following Lemma.

Lemma 3.12. *The symbol expansion of* $[Q_1, a^w]$ *in the* h*-Weyl calculus is of the form*

$$\begin{aligned}
[Q_1, a^w] = & \mathrm{Op}_h^w \Bigg(\left(\frac{ih}{2} \sigma(D_x, D_\xi; D_y, D_\eta) \right) (q_1(x,\xi)a(y,\eta) - q_1(y,\eta)a(x,\xi))|_{x=y,\xi=\eta} \\
& \quad + e(x,\xi) + r_3(x,\xi) \Bigg),
\end{aligned}$$

where e *satisfies*

$$\mathrm{Op}_h^w(e) \ll h \mathrm{Op}_h(\mathsf{H}(a)).$$

Proof. Since everything is in the Weyl calculus, only the odd terms in the exponential composition expansion are non-zero. Hence the h^2 term is zero in the Weyl expansion. Now according to Lemma 2.2 and the standard L^2 continuity theorem for h-pseudodifferential operators, we need to estimate a finite number of derivatives of the error:

(3.12)
$$|\partial^\gamma e_2| \leqslant C h^3 \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R} \\ (y,\eta)\in T^*\mathbb{R}}} \sup_{|\rho|\leqslant M, \rho\in\mathbb{N}^4} \left| \Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^3 q_1(x,\xi) a(y,\eta) \right| .$$

However, since $q_1(x,\xi) = \xi^2 + V_{0,h}(x)$, we have

$$D_x D_\xi q_1 = D_\xi^3 q_1 = 0,$$

so that

$$\begin{aligned}\sigma(D)^3 q_1(x,\xi)a(y,\eta)|_{x=y,\xi=\eta} \\ &= D_x^3 q_1 D_\eta^3 a|_{x=y,\xi=\eta} \\ &= -V_h'''(x)(\tilde{h}/h)^{3m/(m+1)}\Lambda'''((\tilde{h}/h)^{m/(m+1)}\eta) \\ &\quad\times \Lambda((\tilde{h}/h)^{1/(m+1)}y)\chi(y)\chi(\eta) + r_3,\end{aligned}$$

where r_3 is supported in $\{|(x,\xi)| \geqslant \delta_1\}$. Owing to the cutoffs $\chi(y)\chi(\eta)$ in the definition of a (and the corresponding implicit cutoffs in q_1), we only need to estimate this error in compact sets. The derivatives $h^\beta\partial_\eta$ and $h^\alpha\partial_y$ preserve the order of e_2 in h and increase the order in $\tilde{h}$, while the other derivatives lead to higher powers in $h/\tilde{h}$ in the symbol expansion. Hence we need only estimate e_2, as the derivatives satisfy similar estimates.

In order to estimate e_2, we again use conjugation to the 2-parameter calculus, and at some point invoke the 0-Gevrey assumption. We have

$$\|\mathrm{Op}_h^w(e_2)u\| = \|T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1}T_{h,\tilde{h}}u\| \leqslant \|T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1}\|_{L^2\to L^2}\|u\|,$$

by unitarity of $T_{h,\tilde{h}}$. But $T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1} = \mathrm{Op}_{\tilde{h}}^w(e_2\circ\mathcal{B})$ and

$$\begin{aligned}e_2\circ\mathcal{B} &= -h^3V_h'''((h/\tilde{h})^{1/(m+1)}X)(\tilde{h}/h)^{3m/(m+1)}\Lambda'''(\Xi) \\ &\quad\times\Lambda(X)\chi(x)\chi(\xi) + r_3\circ\mathcal{B},\end{aligned}$$

where r_3 is again microsupported away from the critical point (coming from the derivatives on $\chi(x)\chi(\xi)$. We recall that $V_h'''(x) = V'''(x) + W_h'''(x)$, where $W_h(x) = \Gamma(h)f(x/x_0)$. As $f \in \mathcal{C}_c^\infty$, we know that

$$|W_h'''(x)| \leqslant C\Gamma x_0^{-3},$$

and hence

$$\begin{aligned}|h^3(\tilde{h}/h)^{3m/(m+1)}\Lambda'''(\Xi)\Lambda(X)W_h'''((h/\tilde{h})^{1/(m+1)}X)(\chi(x)\chi(\xi)| \\ \leqslant C\Gamma h^{3/(m+1)}\tilde{h}^{3m/(m+1)}x_0^{-3}.\end{aligned}$$

As for V, since $V' \in \mathcal{G}_\tau^0$, for x close to 0 satisfying (in the rescaled coordinates)

$$|X| \geqslant \left(\frac{h}{\tilde{h}}\right)^{-1/(m+1)+\epsilon_1}, \ \ \epsilon_1 > 0,$$

we have

$$\begin{aligned}&|h^{3/(m+1)}\tilde{h}^{3m/(m+1)}V'''((h/\tilde{h})^{1/(m+1)}X)| \\ &\leqslant Ch^{3/(m+1)}\tilde{h}^{3m/(m+1)}\left|\left(\frac{h}{\tilde{h}}\right)^{1/(m+1)}X\right|^{-2\tau}|V_0'((h/\tilde{h})^{1/(m+1)}X)| \\ &\ll h^{2/(m+1)+\gamma}\tilde{h}^{3m/(m+1)}|V_0'((h/\tilde{h})^{1/(m+1)}X)|\end{aligned} \tag{3.13}$$

provided

$$2\tau\epsilon_1 \leqslant \frac{1}{m+1} - \gamma,$$

for $\gamma > 0$ (which of course implies we must have $\gamma < 1/(m+1)$). This can clearly be done for any $\tau < \infty$ by taking $\epsilon_1 > 0$ sufficiently small.

We need to estimate (3.13) in terms of $V_0'(X)\langle\Xi\rangle^{-1-\epsilon_0}$ as (x,ξ) *and* (y,η) vary in (3.12). That means we need to worry about large $|\Xi|$. If $|\Xi| \leqslant \delta_1/2$, say, then (3.13) is trivially bounded by

$$h^{2/(m+1)+\gamma}\tilde{h}^{3m/(m+1)}|V_0'((h/\tilde{h})^{1/(m+1)}X)|\langle\Xi\rangle^{-1-\epsilon_0}.$$

If

$$|\Xi| \geqslant \max\{|X|^{1+\epsilon_0}, \delta_1/2\},$$

then the function $g_1 \geqslant c_{\delta_1}$, so there is nothing to prove in this region. On the other hand, if

$$\frac{\delta_1}{2} \leqslant |\Xi| \leqslant |X|^{1+\epsilon_0},$$

then

$$|\Xi|^{-1} \geqslant |X|^{-1-\epsilon_0},$$

so that

$$\langle\Xi\rangle^{-1-\epsilon_0} \geqslant \langle X\rangle^{-(1+\epsilon_0)^2} \geqslant \left(\frac{h}{\tilde{h}}\right)^{\alpha(1+\epsilon_0)^2}.$$

Then (3.13) is bounded by

$$\begin{aligned} &Ch^{2/(m+1)+\gamma}\tilde{h}^{3m/(m+1)}|V_0'((h/\tilde{h})^{1/(m+1)}X)\left(\frac{h}{\tilde{h}}\right)^{-\alpha(1+\epsilon_0)^2}|\langle\Xi\rangle^{-1-\epsilon_0} \\ &\ll Ch^{1/(m+1)}\tilde{h}^{m/(m+1)}|V_0'((h/\tilde{h})^{1/(m+1)}X)\langle\Xi\rangle^{-1-\epsilon_0}, \end{aligned}$$

provided

$$\frac{(1+\epsilon_0)^2}{m+1} < \frac{1}{m+1} + \gamma,$$

which is possible since we have already determined $\epsilon_0 \ll 1/(m+1)$ and the only restriction on γ was $\gamma < 1/(m+1)$.

On the other hand, we have for

$$|X| \leqslant \left(\frac{h}{\tilde{h}}\right)^{-1/(m+1)+\epsilon_1}, \ \ \epsilon_1 > 0,$$

since $V'''(x) = \mathcal{O}(|x|^\infty)$, then

$$|V'''((h/\tilde{h})^{1/(m+1)}X)| = \mathcal{O}(h^\infty).$$

The error term must be estimated in terms of $h\mathsf{H}(a)$. Recall that $|\Lambda'''(\Xi)| \leqslant C\langle\Xi\rangle^{-1-\epsilon_0}$, so we have shown that the error is always controlled by

$$o(h^{1/(m+1)}\tilde{h}^{m/(m+1)})\langle\Xi\rangle^{-1-\epsilon_0}|\Lambda(X)V_{0,h}'((h/\tilde{h})^{1/(m+1)}X)| + \mathcal{O}(h^\infty) \ll h\mathsf{H}(a).$$

□

Finally, we are able to put things together. Let $v = \varphi^w u$, with φ chosen to have support inside the set where $\chi(x)\chi(\xi) = 1$; thus the terms r and r_3 above are supported away from the support of φ. Then Lemma 3.12 yields

$$\begin{aligned} &i\langle[Q_1 - z, a^w]v, v\rangle \\ &\quad = h\langle \mathrm{Op}_h^w(\mathsf{H}(a))v, v\rangle + \langle \mathrm{Op}_h^w(e_2)u, u\rangle \\ &\quad \geqslant C\Gamma x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}\|v\|^2 \\ &\quad = Ch^{2m^2/(m^2-1)+3\epsilon_0/(m^2-m)}\tilde{h}^{m/(m+1)-3\epsilon_0/(m^2-m)}x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}\|v\|^2, \end{aligned}$$

for $\tilde{h}$ sufficiently small. Here we have used the previously computed value of Γ. On the other hand, we certainly have

$$\left|\langle [Q_1 - z, a^w]v, v\rangle\right| \leqslant C\|(Q_1 - z)v\|\|v\|,$$

hence

$$\|(Q_1 - z)v\| \geqslant C\Gamma x_0^{-3-\epsilon_0+(1+\epsilon_0)/m}\|v\|.$$

We need yet compare $\widetilde{Q}$ to Q_1:

$$\begin{aligned}
&\|v\| \\
&\leqslant C\Gamma^{-1}x_0^{3+\epsilon_0-(1+\epsilon_0)/m}\|(Q_1 - z)v\| \\
&\leqslant C\Gamma^{-1}x_0^{3+\epsilon_0-(1+\epsilon_0)/m}\left(\left\|(\widetilde{Q} - z)v\right\| + \|(V_{0,h} - V_0)v\|\right) \\
&\leqslant C\Gamma^{-1}x_0^{3+\epsilon_0-(1+\epsilon_0)/m}\left(\left\|(\widetilde{Q} - z)v\right\| + \Gamma(h)\|v\|\right) \\
&\leqslant C\Gamma^{-1}x_0^{3+\epsilon_0-(1+\epsilon_0)/m}\left\|(\widetilde{Q} - z)v\right\| + o(1)\|v\|
\end{aligned}$$

provided that again ϵ_0 is sufficiently small and m is sufficiently large. Then the term with $\|v\|$ can be moved to the left hand side to get (now freezing $\tilde{h}$ small and positive)

$$\begin{aligned}
\|v\| &\leqslant C\Gamma^{-1}x_0^{3+\epsilon_0-(1+\epsilon_0)/m}\left\|(\widetilde{Q} - z)v\right\| \\
&\leqslant Ch^{-2m^2/(m^2-1)-(1+\epsilon_0)/(m+1)}\left\|(\widetilde{Q} - z)v\right\| \\
&= Ch^{-2-\eta}\left\|(\widetilde{Q} - z)v\right\|
\end{aligned}$$

for $\eta = \mathcal{O}(m^{-1})$. This is (3.4).

Lastly, we show how to modify the preceding argument in the case of Proposition 3.8. The main point is that the nonlinear rescaling in Γ (as part of λ) allows us to use that $\Gamma^{1/(m+1)} \gg \Gamma$. The first step is to modify the function f and subsequently W_h and $V_{0,h}$. Since $V_0(x) \equiv 1$ on an interval $x \in [-a, a]$, with $\pm V_0'(x) < 0$ for $\pm x > a$, we choose the point $x_0 > 0$ so that

$$-xV_0'(x) \geqslant \frac{h}{\varpi(h)}, \quad a + x_0 \leqslant x \leqslant a + \epsilon,$$

and similarly for $-a - \epsilon \leqslant x \leqslant -a - x_0$. Again we can assume that $|x_0 - a| = o(1)$. Then choose $f \in \mathcal{C}_c^\infty(\mathbb{R}) \cap \mathcal{G}_\tau^0$ for some $\tau < \infty$, with $f(x) = 1 - \frac{1}{2m}x^{2m}$ for $|x| \leqslant a + x_0$, and $f'(x) \leqslant 0$ or $x \geqslant 0$, $\operatorname{supp} f \subset [-a - 2x_0, a + 2x_0]$, satisfying

$$|\partial_x^k f| \leqslant C_k |x_0|^{-k}.$$

For our next parameter, set $\widetilde{\Gamma}(h) = c_0\Gamma(h)$ for a small constant $c_0 > 0$ to be determined, and $\Gamma(h)$ the parameter computed in the case of the isolated infinitely degenerate maximum. As before then we take

$$W_h(x) = \widetilde{\Gamma}(h)f(x),$$

and let

$$V_{0,h}(x) = V_0(x) + W_h(x)$$

We then follow the same arguments as in the proofs of Proposition 3.2 and 3.6, noting that the "smallness" assumption on the support of the microlocal cutoff φ in the x direction was to control lower order terms in Taylor expansions. As the

function V_0 is constant and $f(x) = 1 - x^{2m}$ on $[-a, a]$, the smallness assumption translates into a small neighbourhood around $[-a, a]$. Hence in Proposition 3.8 we have assumed that $\operatorname{supp} \varphi \subset [-a - \epsilon, a + \epsilon]$. All of the error terms are treated similarly to the preceding proof. The only changes to check are that, since f is no longer a function of x/x_0, and $\widetilde{\Gamma}(h) = c_0 \Gamma$, we need to solve (in the previous notation):

$$h(h/\tilde{h})^{(m-1)/(m+1)} \widetilde{\Gamma}^{1/m} (h/\tilde{h})^{\alpha(1+\epsilon_0)(m-1)/m} \tilde{h}^{2(m-1)/m} \gg \widetilde{\Gamma},$$

or

$$h(h/\tilde{h})^{(m-1)/(m+1)} (h/\tilde{h})^{\alpha(1+\epsilon_0)(m-1)/m} \tilde{h}^{2(m-1)/m} \gg (c_0 \Gamma)^{(m-1)/m},$$

which is true with our previous choice of Γ, provided $c_0 > 0$ is sufficiently small and independent of h.

As previously, we then have

$$\|(Q_1 - z)v\| \geqslant C^{-1} c_0^{1/m} h^{2m^2/(m^2-1)+3\epsilon_0/(m^2-m)} \tilde{h}^{m/(m+1)-3\epsilon_0/(m^2-m)} \|v\|.$$

In order to save some space, let us denote

$$\omega(h) = h^{2m^2/(m^2-1)+3\epsilon_0/(m^2-m)} \tilde{h}^{m/(m+1)-3\epsilon_0/(m^2-m)}.$$

Comparing $\widetilde{Q}$ to Q_1 now yields:

$$\begin{aligned} \|v\| &\leqslant C c_0^{-1/m} \omega(h)^{-1} \|(Q_1 - z)v\| \\ &\leqslant C c_0^{-1/m} \omega(h)^{-1} \left(\left\| (\widetilde{Q} - z)v \right\| + \|(V_{0,h} - V_0)v\| \right) \\ &\leqslant C c_0^{-1/m} \omega(h)^{-1} \left(\left\| (\widetilde{Q} - z)v \right\| + C c_0 \omega(h) \|v\| \right) \\ &\leqslant C c_0^{-1/m} \omega(h)^{-1} \left\| (\widetilde{Q} - z)v \right\| + C c_0^{(m-1)/m} \|v\|. \end{aligned}$$

Freezing $\tilde{h} > 0$ and $c_0 > 0$ sufficiently small, the term with $\|v\|$ can be moved to the left hand side to get

$$\|v\| \leqslant C h^{-2-\eta} \left\| (\widetilde{Q} - z)v \right\|$$

with $\eta = \mathcal{O}(m^{-1})$ once again. This is (3.5).

□

3.5. **Infinitely degenerate and cylindrical inflection transmission trapping.** In this subsection, we study the microlocal spectral theory in a neighbourhood of infinitely degenerate and cylindrical inflection transmission trapping. This is very similar to Subsection 3.4, but now the potential is assumed to be monotonic in a neighbourhood of the critical value.

We begin with the case where the potential has an isolated infinitely degenerate critical point of inflection transmission type. As in the previous subsection, we write $V(x) = A^{-2}(x) + h^2 V_1(x)$ and denote $V_0(x) = A^{-2}(x)$ to be the principal part of the potential. Let us assume the point $x = 1$ is an infinitely degenerate inflection point, so that locally near $x = 1$, the potential takes the form

$$V_0(x) \sim C_1^{-1} - (x-1)^\infty,$$

where $C_1 > 1$. Of course the constant is arbitrary (chosen to again agree with those in [CM13]). Let us assume that our potential satisfies $V_0'(x) \leqslant 0$ near $x = 1$,

with $V_0'(x) < 0$ for $x \neq 1$ so that the critical point $x = 1$ is isolated. The next Proposition says that in this case the microlocal resolvent is bounded by $\mathcal{O}(h^{-2-\eta})$ for any $\eta > 0$. Let

$$\widetilde{Q} = (hD_x)^2 + V(x) - z.$$

Proposition 3.13. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x-1,\xi)| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C = C_{\epsilon,\eta} > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_\epsilon h^{2+\eta}\|\varphi^w u\|, \;\; z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon]. \tag{3.14}$$

On the other hand, if $V_0'(x) \equiv 0$ on an interval, say $x-1 \in [-a,a]$ with $V_0'(x) < 0$ for $x - 1 < -a$ and $x - 1 > a$, we do not expect anything better than Proposition 3.13. The next Proposition says that this is exactly what we do get. To fix an energy level, assume $V_0 \equiv C_1^{-1}$ on $[-a,a]$. We again write

$$\widetilde{Q} = (hD_x)^2 + V(x) - z.$$

Proposition 3.14. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|x-1| \leqslant a+\epsilon,\ |\xi| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C = C_{\epsilon,\eta} > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_\epsilon h^{2+\eta}\|\varphi^w u\|, \;\; z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon]. \tag{3.15}$$

Proof. The proof of these Propositions is again very similar, so we put them together. We will first prove Proposition 3.13, and then point out how the proof must be modified to get Proposition 3.14.

The idea of the proof of Proposition 3.13 (and indeed Proposition 3.14) is to "round off the corners" in an h-dependent fashion to obtain a *finitely degenerate* inflection point, and then mimic the proof of Proposition 3.4.

Choose a point $x_0 = x_0(h) > 0$ and $\epsilon > 0$ such that x_0 is the smallest number so that

$$-V_0'(x) \geqslant \frac{h}{\varpi(h)}, \;\; x_0 \leqslant |x-1| \leqslant \epsilon,$$

where $\varpi(h)$ will be determined later. Similar considerations apply to choosing the parameters here as in the previous subsection, but since we wrote that in excruciating detail, we will leave out one or two details in this subsection. Fix $m_2 \geqslant 1$, and choose also an odd function $f \in \mathcal{C}_c^\infty([-2,2]) \cap \mathcal{G}_\tau^0$ for some $\tau < \infty$, with $f(x) = -(x)^{2m_2+1}/(2m_2+1)$ for $|x| \leqslant 1$ and $f(x), f'(x) \leqslant 0$ for $0 \leqslant x \leqslant 2$. For another parameter $\Gamma(h)$ to be determined, let

$$W_h(x) = \Gamma(h) f((x-1)/x_0),$$

and let

$$V_{0,h}(x) = V_0(x) + W_h(x)$$

and

$$V_h(x) = V(x) + W_h(x)$$

(see Figure 2). The parameter $\Gamma(h)$ will be seen to be a constant multiple of $h^{2+\eta}$, where $\eta > 0$, $\eta = \mathcal{O}(m_2^{-1})$ as $m_2 \to \infty$ in the case of Proposition 3.13. As in the previous subsection, $\Gamma(h)$ will be a small constant times this power of h in the case of Proposition 3.14. By construction,

$$|V_0(x) - V_{0,h}(x)| \leqslant |W_h| \leqslant \Gamma(h).$$

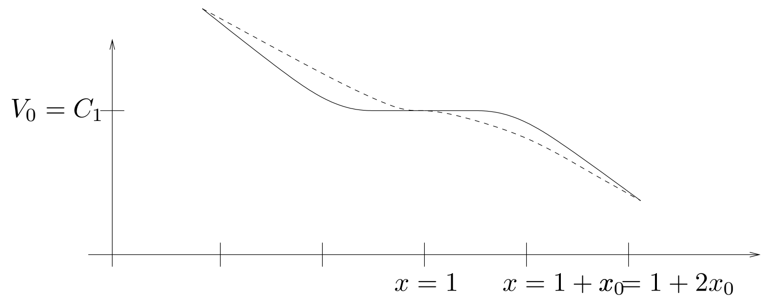

FIGURE 2. The potential V_0 and the modified potential $V_{0,h}$ (in dashed).

Let $Q_1 = (hD)^2 + V_h$ with symbol $q_1 = \xi^2 + V_h$. The Hamilton vector field H associated to the symbol q_1 is given by

$$\begin{aligned}\mathsf{H} &= 2\xi\partial_x - V_h'\partial_\xi \\ &= 2\xi\partial_x - \left(\frac{\Gamma(h)}{x_0}f'((x-1)/x_0) + V_0'(x) + h^2 V_1'(x)\right)\partial_\xi.\end{aligned}$$

We will consider a commutant localizing in this region and singular at the critical point in a controlled way: we introduce new variables

$$\Xi = \frac{\xi}{(h/\tilde{h})^\beta}, \quad X - 1 = \frac{x-1}{(h/\tilde{h})^\alpha},$$

with $\alpha, \beta > 0$, $\alpha = 1/(m_2+1)$, and $\alpha+\beta = 1$ so that we may use the two-parameter calculus.

We remark that in the new "blown-up" coordinates Ξ, X,

$$\mathsf{H} = (h/\tilde{h})^{\beta-\alpha}\big(2\Xi\partial_X - (h/\tilde{h})^{\alpha-2\beta}V_h'((h/\tilde{h})^\alpha(X-1)+1)\partial_\Xi\big) \tag{3.16}$$

Now fix $\epsilon_0 > 0$ and set

$$\Lambda_1(s) = \int_0^s \langle s'\rangle^{-1-\epsilon_0}\, ds'$$

and

$$\Lambda_2(s) = 1 + \int_{-\infty}^s \langle s'\rangle^{-1-\epsilon_0}\, ds'.$$

Λ_1 is a bounded symbol which looks like s near 0, and Λ_2 is a bounded symbol with positive derivative for s near 0, and $\Lambda_2 \geqslant 1$ everywhere.

We introduce the singular symbol

$$\begin{aligned}a(x,\xi;h) &= \Lambda_1(\Xi)\Lambda_2(X-1)\chi(x-1)\chi(\xi) \\ &= \Lambda_1(\xi/(h/\tilde{h})^\beta)\Lambda_2((x-1)/(h/\tilde{h})^\alpha)\chi(x-1)\chi(\xi),\end{aligned}$$

where $\chi(s)$ is a cutoff function equal to 1 for $|s| \leqslant \delta_1$ and 0 for $s \geqslant 2\delta_1$ (δ_1 will be chosen shortly). Then a is bounded since we have restricted the domain of integration to $|(x-1,\xi)| \leqslant \delta_1$. Further, a satisfies the symbolic estimates:

$$\left|\partial_X^{\vec{\alpha}}\partial_\Xi^{\vec{\beta}} a\right| \leqslant C_{\vec{\alpha},\vec{\beta}}.$$

(Recall that $x-1=(h/\tilde{h})^{\alpha}(X-1)$ and $\xi=(h/\tilde{h})^{\beta}\Xi$.) Using (3.16), it is simple to compute

$$\begin{aligned}\mathsf{H}(a) =&(h/\tilde{h})^{\beta-\alpha}\chi(x-1)\chi(\xi)\big(2\Lambda_1(\Xi)\langle X-1\rangle^{-1-\epsilon_0}\Xi\\ &-(h/\tilde{h})^{\alpha-2\beta}V_h'((h/\tilde{h})^{\alpha}(X-1)+1)\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1)\big)+r\\ =:&(h/\tilde{h})^{\beta-\alpha}g+r\end{aligned}\tag{3.17}$$

with

$$\operatorname{supp} r\subset\{|x-1|>\delta_1\}\cup\{|\xi|>\delta_1\}$$

(r comes from terms involving derivatives of $\chi(x-1)\chi(\xi)$).

For $|X-1|\leqslant(h/\tilde{h})^{-\alpha}x_0$ we have

$$\begin{aligned}&-(h/\tilde{h})^{\alpha-2\beta}V_h'((h/\tilde{h})^{\alpha}(X-1)+1)\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1)\\ &\quad=\Gamma(h)x_0^{-(2m_2+1)}(h/\tilde{h})^{\alpha(2m_2+1)-2\beta}(X-1)^{2m_2}\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1)+g_2,\end{aligned}$$

with

$$g_2=-(h/\tilde{h})^{\alpha-2\beta}(V_0'+h^2V_1')((h/\tilde{h})^{\alpha}(X-1)+1)\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1).$$

Let us denote by g_1 the part of g obtained in this fashion, microlocally in $\{|X-1|\leqslant(h/\tilde{h})^{-\alpha}x_0\}$:

$$\begin{aligned}g_1&=g-g_2\\ &=2\Lambda_1(\Xi)\langle X-1\rangle^{-1-\epsilon_0}\Xi\\ &\quad+\Gamma(h)x_0^{-(2m_2+1)}(h/\tilde{h})^{\alpha(2m_2+1)-2\beta}(X-1)^{2m_2}\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1).\end{aligned}$$

For $|X-1|\leqslant(h/\tilde{h})^{-\alpha}x_0$ and $|\Xi|\leqslant(h/\tilde{h})^{-\beta}\delta_1$ consider

$$\begin{aligned}g_1=&2\Lambda_1(\Xi)\Xi\langle X-1\rangle^{-1-\epsilon_0}\\ &+\frac{\Gamma(h)}{x_0^{2m_2+1}}(h/\tilde{h})^{\alpha(2m_2+1)-2\beta}(X-1)^{2m_2}\Lambda_2(X-1)\langle\Xi\rangle^{-1-\epsilon_0}\\ =&2\Lambda_1(\Xi)\Xi\langle X-1\rangle^{-1-\epsilon_0}\\ &+\frac{\Gamma(h)}{x_0^{2m_2+1}}(h/\tilde{h})^{\alpha}(X-1)^{2m_2}\Lambda_2(X-1)\langle\Xi\rangle^{-1-\epsilon_0},\end{aligned}$$

where we have used $\alpha=1/(m_2+1)$. Continuing, and rescaling using the L^2-unitary rescaling

$$X'-1=\lambda(X-1),\;\;\Xi'=\lambda^{-1}\Xi,$$

we get

$$\begin{aligned}g_1=&\lambda^2\Big(2\lambda^{-1}\Lambda_1(\Xi)(\lambda^{-1}\Xi)\langle X-1\rangle^{-1-\epsilon_0}\\ &+\lambda^{-2-2m_2}\Gamma x_0^{-2m_2-1}(h/\tilde{h})^{\alpha}\Lambda_2(X-1)(\lambda(X-1))^{2m_2}\langle\Xi\rangle^{-1-\epsilon_0}\Big)\\ =&\lambda^2\Big(2\lambda^{-1}\Lambda_1(\lambda\Xi')\Xi'\big\langle\lambda^{-1}(X'-1)\big\rangle^{-1-\epsilon_0}\\ &+\lambda^{-2-2m_2}\Gamma x_0^{-2m_2-1}(h/\tilde{h})^{\alpha}\Lambda_2(\lambda^{-1}(X'-1))(X'-1)^{2m_2}\langle\lambda\Xi'\rangle^{-1-\epsilon_0}\Big).\end{aligned}$$

As in the previous subsection, the parameter $\lambda>0$ will be seen to be a small h-dependent parameter chosen to optimize lower bounds on g_1 amongst several different regions.

The error term g_2 is the term in the expansion of g coming from V' rather than W_h'. We will deal with g_2 in due course. We are now microlocalized on a set where

$$|X'-1| \leqslant \lambda(h/\tilde{h})^{-\alpha}x_0, \;\; |\Xi'| \leqslant \lambda^{-1}(h/\tilde{h})^{-\beta}\delta_1,$$

and will be quantizing in the $\tilde{h}$-Weyl calculus, so we need symbolic estimates on these sets.

If

$$|X'-1| \leqslant \lambda\delta_1, \text{ and } |\Xi'| \leqslant \lambda^{-1}\delta_1,$$

and $\delta_1 > 0$ is sufficiently small, then $\Lambda_1(\lambda\Xi') \sim \lambda\Xi'$ and $\Lambda_2(\lambda^{-1}(X'-1)) \sim 1$, so that g_1 is bounded below by a multiple of

$$\min\{\lambda^2, \lambda^{-2-2m_2}\Gamma(h/\tilde{h})^{\alpha}\}((\Xi')^2 + (X'-1)^{2m_2}). \tag{3.18}$$

Hence the $\tilde{h}$-quantization of g_1 is bounded below by this minimum value times $\tilde{h}^{2m_2/(m_2+1)}$ on this set (using [CW11, Lemma A.2]).

Now on the complementary set, if $|\lambda\Xi'| \geqslant \max\left(\left|\lambda^{-1}(X'-1)\right|^{1+\epsilon_0}, (\delta_1/2)^{1+\epsilon_0}\right)$ then

$$\begin{aligned} g_1 &\geqslant c\lambda^2\lambda^{-1}\Lambda_1(\lambda\Xi')\Xi'\left\langle\lambda^{-1}(X'-1)\right\rangle^{-1-\epsilon_0} \\ &\geqslant c\lambda \operatorname{sgn}(\Xi')\Xi'|\lambda\Xi'|^{-1} \\ &\geqslant c_0 \end{aligned}$$

for some $c_0 > 0$.

If $\left|\lambda^{-1}(X'-1)\right|^{1+\epsilon_0} \geqslant \max\left(|\lambda\Xi'|, (\delta_1/2)^{1+\epsilon_0}\right)$, we have two regions to consider. The first, if $|\lambda\Xi'| \leqslant (\delta_1/2)^{1+\epsilon_0}$ (and using that $|\lambda^{-1}X'| \leqslant (h/\tilde{h})^{-\alpha}x_0$ in this region), then

$$\begin{aligned} g_1 &\geqslant c\lambda^2\Big((\Xi')^2(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0} \\ &\qquad + \lambda^{-2-2m_2}\Gamma x_0^{-2m_2-1}(h/\tilde{h})^{\alpha}(X'-1)^{2m_2}\Big) \qquad (3.19) \\ &\geqslant \min\{\lambda^2(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0}, \lambda^{-2m_2}\Gamma x_0^{-2m_2-1}(h/\tilde{h})^{\alpha}\} \\ &\qquad \times((\Xi')^2 + (X'-1)^{2m_2}). \qquad (3.20) \end{aligned}$$

We optimize this by setting the two terms in the minimum equal:

$$\lambda^2(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0} = \lambda^{-2m_2}\Gamma x_0^{-2m_2-1}(h/\tilde{h})^{\alpha},$$

or

$$\lambda^{2+2m_2} = \Gamma(h/\tilde{h})^{-\alpha\epsilon_0}x_0^{-2m_2+\epsilon_0},$$

which yields in turn the lower bound

$$\begin{aligned} &\lambda^2(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0} \\ &\quad = \Gamma^{1/(m_2+1)}(h/\tilde{h})^{-\alpha\epsilon_0/(m_2+1)+\alpha(1+\epsilon_0)}x_0^{(-2m_2+\epsilon_0)/(m_2+1)-1-\epsilon_0}. \end{aligned}$$

Then according to [CW11, Lemma A.2], the $\tilde{h}$-quantization of (3.20) is bounded below by

$$\Gamma^{1/(m_2+1)}(h/\tilde{h})^{-\alpha\epsilon_0/(m_2+1)+\alpha(1+\epsilon_0)}x_0^{(-2m_2+\epsilon_0)/(m_2+1)-1-\epsilon_0}\tilde{h}^{2m_2/(m_2+1)}.$$

On the other hand, if $|\lambda^{-1}(X'-1)|^{1+\epsilon_0} \geqslant |\lambda \Xi'| \geqslant (\delta_1/2)^{1+\epsilon_0}$, then

$$\begin{aligned} g_1 &\geqslant c\operatorname{sgn}(\Xi')\lambda\Xi'(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0} \\ &\geqslant c(h/\tilde{h})^{\alpha(1+\epsilon_0)}x_0^{-1-\epsilon_0}. \end{aligned} \tag{3.21}$$

We now again take the worst lower bound for (3.18)-(3.21) to get for a function u with h-wavefront set localized in

$$|(X'-1)| \leqslant \lambda(h/\tilde{h})^{-\alpha}x_0, \ |\Xi'| \leqslant \lambda^{-1}(h/\tilde{h})^{-\beta}\delta_1,$$

$$\begin{aligned} &\left\langle \operatorname{Op}_{\tilde{h}}(g_1)u, u\right\rangle \\ &\quad \geqslant c\Gamma^{1/(m_2+1)}h^{(1+\epsilon_0)/(m_2+1)-\epsilon_0/(m_2+1)^2} \\ &\quad\quad \times \tilde{h}^{2-(3+\epsilon_0)/(m_2+1)+\epsilon_0/(m_2+1)^2}x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)}\|u\|^2. \end{aligned}$$

Here we have used that $\alpha = 1/(m_2+1)$.

On the other hand, if $|\lambda^{-1}(X'-1)| \geqslant (h/\tilde{h})^{-\alpha}x_0$, we use the assumed lower bound on V_0' to estimate g from below. Examining the potential terms, we have

$$\begin{aligned} &-(h/\tilde{h})^{\alpha-2\beta}V_h'((h/\tilde{h})^{\alpha}X)\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1)\big) \\ &\quad = -(h/\tilde{h})^{\alpha-2\beta}V'((h/\tilde{h})^{\alpha}X)\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1)\big) + g_3 \\ &\quad \geqslant C(h/\tilde{h})^{\alpha-2\beta}\frac{h}{\varpi(h)}(h/\tilde{h})^{(1+\epsilon_0)\beta} + g_3 \\ &\quad = C\frac{h^{2\alpha+\epsilon_0\beta}\tilde{h}^{\beta-\alpha-\epsilon_0\beta}}{\varpi(h)} + g_3, \end{aligned} \tag{3.22}$$

assuming that $h/\varpi(h) \gg h^2$ so that V_0' controls h^2V_1' (this will be verified later). The error $g_3 \geqslant 0$ comes from using V' in the expansion of g rather than W_h'.

We now deal with the (nearly) positive error terms g_2 and g_3.

Lemma 3.15. *The error terms g_2 and g_3 are semi-bounded below in the following sense: if $u(X)$ has wavefront set localized in*

$$\{|X-1| \leqslant \epsilon(h/\tilde{h})^{-\alpha}, \ |\Xi| \leqslant \epsilon(h/\tilde{h})^{-\beta}\},$$

then for any $\delta > 0$ and $N > 0$,

$$\left\langle \operatorname{Op}_{\tilde{h}}(g_j)u, u\right\rangle \geqslant -C_N h^{(N+1)\alpha-2\beta-\delta}\tilde{h}^{2\beta-\alpha}\|u\|^2,$$

for $j = 2, 3$.

Proof. We prove the relevant bounds for $x \geqslant 1$. The analysis for $x \leqslant 1$ is similar. For g_2, for $N > 0$ large, and $\delta > 0$ small, choose $1 < x_1 < x_2 = 1 + o(1)$ satisfying

$$-V_0'(x_1) = h^{N\alpha}$$

and

$$-V_0'(x_2) = h^{N\alpha-\delta}.$$

As usual, since $V_0'(x) = \mathcal{O}((x-1)^\infty)$, the points x_j, $j = 1, 2$ satisfy $x_j - 1 \gg h^{\delta_1}$ for any $\delta_1 > 0$. As before, the 0-Gevrey condition also implies $|x_2 - x_1| \gg h^{\delta_1}$ for any $\delta_1 > 0$ as well.

Now let $\psi(x)$ be a smooth function, $\psi \geqslant 0$, $\psi(x) \equiv 1$ on $[1, x_1]$ with $\psi(x) = 0$ for $x \geqslant x_2$. Assume also that $|\partial_x^k\psi| \leqslant C_k|x_2 - x_1|^{-k} = o(h^{-k\delta_1})$ for any $\delta_1 > 0$. Let $\tilde{\psi}(X) = \psi((h/\tilde{h})^{\alpha}X)$ so that

$$|\partial_X^k\tilde{\psi}| \leqslant C_k(h/\tilde{h})^{\alpha k}|x_2 - x_1|^{-k} = o(h^{k(\alpha-\delta_1)}\tilde{h}^{-\alpha k}).$$

We have

$$\begin{aligned}
&\left\langle \mathrm{Op}_{\tilde{h}}(g_2)u, u\right\rangle \\
&\quad = \left\langle \mathrm{Op}_{\tilde{h}}(g_2)(1-\tilde{\psi})u, (1-\tilde{\psi})u\right\rangle + \left\langle \mathrm{Op}_{\tilde{h}}(g_2)\tilde{\psi}u, \tilde{\psi}u\right\rangle \\
&\qquad + 2\left\langle \mathrm{Op}_{\tilde{h}}(g_2)\tilde{\psi}u, (1-\tilde{\psi})u\right\rangle.
\end{aligned}$$

We estimate each term separately.

On the support of $1-\tilde{\psi}$ (recalling once again that we are restricting our attention to $x \geqslant 1$), we have $(h/\tilde{h})^\alpha X \geqslant x_1$ so that in this region we can once again appeal to the 0-Gevrey condition to control V_1'. As discussed previously, V_1 consists of quotients of derivatives of the function A with powers of A. As A is bounded above and below for x small, we again use the 0-Gevrey condition to write for any $\delta_1 > 0$ and for some $s, \tau < \infty$,

$$\begin{aligned}
h^2|V_1'((h/\tilde{h})^\alpha(X-1)+1)| &\leqslant Ch^2|x_1|^{-s\tau}|A'((h/\tilde{h})^\alpha(X-1)+1)| \\
&\leqslant Ch^{2-s\tau\delta_1}|V_0'((h/\tilde{h})^\alpha(X-1)+1)|,
\end{aligned}$$

and similarly for a finite number of derivatives of V_1. Taking $\delta_1 > 0$ sufficiently small, we see that on the support of $1-\tilde{\psi}$, V_0' controls h^2V_1'. That is, for $h > 0$ sufficiently small,

$$\begin{aligned}
&\left\langle \mathrm{Op}_{\tilde{h}}(g_2)(1-\tilde{\psi})u, (1-\tilde{\psi})u\right\rangle \\
&\quad = -(h/\tilde{h})^{\alpha-2\beta}\left\langle \mathrm{Op}_{\tilde{h}}((V_0'+h^2V_1')\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1))(1-\tilde{\psi})u, (1-\tilde{\psi})u\right\rangle \\
&\quad \geqslant -\frac{1}{2}(h/\tilde{h})^{\alpha-2\beta}\left\langle \mathrm{Op}_{\tilde{h}}(V_0'\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1))(1-\tilde{\psi})u, (1-\tilde{\psi})u\right\rangle.
\end{aligned}$$

Here to save space (and since it will be integrated out anyway) we suppressed the argument of V_0 and V_1; both functions are understood to be evaluated at $((h/\tilde{h})^\alpha(X-1)+1)$. Then in this same region we have:

$$\begin{aligned}
-(h/\tilde{h})^{\alpha-2\beta}\Lambda_2(X-1)V_0'((h/\tilde{h})^\alpha(X-1)+1)\left\langle\Xi\right\rangle^{-1-\epsilon_0} \\
= (h/\tilde{h})^{\alpha-2\beta}h^{N\alpha}A(X,h,\tilde{h})\left\langle\Xi\right\rangle^{-1-\epsilon_0}
\end{aligned}$$

where A is a symbol bounded below by a positive constant. On the set where $A\left\langle\Xi\right\rangle^{-1-\epsilon_0} \geqslant 1$, this operator is bounded below, while on the complement, we use the Sharp Gårding inequality to get for any $\delta_1 > 0$

$$\left\langle \mathrm{Op}_{\tilde{h}}(g_2)(1-\tilde{\psi})u, (1-\tilde{\psi})u\right\rangle \geqslant -C_{\delta_1}\tilde{h}h^{N\alpha+2\alpha-2\beta-\delta_1}\tilde{h}^{2\beta-2\alpha}\|(1-\tilde{\psi})u\|^2.$$

For the remaining two terms, on the support of $\tilde{\psi}$, we have $1 \leqslant (h/\tilde{h})^\alpha X \leqslant x_2$, so that in order to estimate V_1', we need to estimate a finite number of derivatives of A from above. But we know that for $k \geqslant 1$, $|\partial_x^k A|$ is an increasing function for $x \geqslant 1$ in a small neighbourhood. Hence we can estimate the size of V_1' by estimating it at x_2. For this, we once again use the 0-Gevrey assumption to get for any $\delta_1 > 0$ and for some $s, \tau < \infty$

$$\begin{aligned}
h^2|V_1'((h/\tilde{h})^\alpha(X-1)+1)| &\leqslant Ch^2|x_2|^{-s\tau}|V_0'(x_2)| \\
&\leqslant Ch^{2-\delta_1 s\tau}h^{Nm/(m+1)-\delta},
\end{aligned}$$

by our choice of x_2. This shows that on the support of $\tilde{\psi}$, $h^2 V_1'$ is controlled by a large power of h. Then in this region

$$\begin{aligned} g_2 &= -(h/\tilde{h})^{\alpha-2\beta}\Lambda_2(X)V'((h/\tilde{h})^{\alpha}X)\langle\Xi\rangle^{-1-\epsilon_0} \\ &= (h/\tilde{h})^{\alpha-2\beta}h^{N\alpha-\delta}A_1(X,h,\tilde{h})\langle\Xi\rangle^{-1-\epsilon_0}, \end{aligned}$$

where A_1 is a function satisfying

$$|\partial_X^k A_1| \leqslant C_{k,\delta_1}(h^{\alpha-\delta_1}\tilde{h}^{-\alpha})^k.$$

Hence if $\delta_1 < \alpha$,

$$\left\langle \mathrm{Op}_{\tilde{h}}(g_2)\tilde{\psi}u, \tilde{\psi}u \right\rangle = \mathcal{O}(h^{(N+1)\alpha-2\beta-\delta}\tilde{h}^{2\beta-\alpha})\|u\|^2,$$

and similarly

$$\left\langle \mathrm{Op}_{\tilde{h}}(g_2)\tilde{\psi}u, \tilde{\psi}u \right\rangle = \mathcal{O}(h^{(N+1)\alpha-2\beta-\delta}\tilde{h}^{2\beta-\alpha})\|u\|^2.$$

As in Lemma 3.11, the proof for g_3 is the same.

□

We are now in position again to fix some of the parameters. We start with Γ, which we again want to be much smaller than our computed lower bound on $h\mathrm{Op}_h(\mathsf{H}(a))$. We need to solve

$$\begin{aligned} & h(h/\tilde{h})^{(m_2-1)/(m_2+1)}\Gamma^{1/(m_2+1)}h^{(1+\epsilon_0)/(m_2+1)-\epsilon_0/(m_2+1)^2} \\ & \quad \times \tilde{h}^{2-(3+\epsilon_0)/(m_2+1)+\epsilon_0/(m_2+1)^2}x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)} \\ & \gg \Gamma \end{aligned}$$

Again, $m_2 > 0$ will be large, $\epsilon_0 > 0$ is small, and $x_0 > 0$ is $o(1)$, so it suffices to solve

$$\begin{aligned} & h^{2-(1-\epsilon_0)/(m_2+1)-\epsilon_0/(m_2+1)^2} \\ & \quad \times \tilde{h}^{2-(m_2-1)/(m_2+1)-(3+\epsilon_0)/(m_2+1)+\epsilon_0/(m_2+1)^2} \\ & \quad\quad = \Gamma^{m_2/(m_2+1)}, \end{aligned}$$

or

$$\Gamma = h^{2+1/m_2+\epsilon_0/m_2-\epsilon_0/m_2(m_2+1)}\tilde{h}^{1-\epsilon_0/m_2+\epsilon_0/m_2(m_2+1)}$$

Then for this value of Γ, our lower bound on $h\mathrm{Op}_h(\mathsf{H}(a))$ is

$$\begin{aligned} & \Gamma x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)} \\ & \quad = h^{2+2/m_2-(1-\epsilon_0)/m_2-\epsilon_0/m_2(m_2+1)} \\ & \quad\quad \times \tilde{h}^{2+2/m_2-(m_2-1)/m_2-(3+\epsilon_0)/m_2+\epsilon_0/m_2(m_2+1)} \\ & \quad\quad \times x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)}. \end{aligned}$$

We again observe that in this case, the exponent of h is $2+\mathcal{O}(m_2^{-1})$, which can be made smaller than $2+\eta$ for any $\eta > 0$.

We can now choose the parameter $\varpi(h)$ as well, again by matching:

$$\begin{aligned}
&\frac{h^{2\alpha+\epsilon_0\beta}\tilde{h}^{\beta-\alpha-\epsilon_0\beta}}{\varpi(h)} \\
&\quad = \Gamma^{1/(m_2+1)} h^{(1+\epsilon_0)/(m_2+1)-\epsilon_0/(m_2+1)^2} \\
&\qquad \times \tilde{h}^{2-(3+\epsilon_0)/(m_2+1)+\epsilon_0/(m_2+1)^2} x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)} \\
&\quad = h^{\frac{3m_2+1}{m_2^2+m_2}+\frac{\epsilon_0}{m_2+1}} \\
&\qquad \times \tilde{h}^{2-\frac{2}{m_2+1}-\frac{\epsilon_0}{m_2+1}} \\
&\qquad \times x_0^{-3-\epsilon_0+\frac{(2+\epsilon_0)}{(m_2+1)}},
\end{aligned}$$

or

$$\varpi(h) = h^{\gamma_1}\tilde{h}^{\gamma_2} x_0^{3+\epsilon_0-(2+\epsilon_0)/(m_2+1)},$$

where

$$\gamma_1 = -\frac{1}{m_2} + \epsilon_0\left(\frac{m_2-1}{m_2+1}\right)$$

and

$$\gamma_2 = -1 - \epsilon_0\left(\frac{m_2-1}{m_2+1}\right).$$

All told, we have shown for a function $u(X)$ with semiclassical wavefront set localized in a set $\{|X-1| \leqslant \epsilon(h/\tilde{h})^{-\alpha}, |\Xi| \leqslant \epsilon(h/\tilde{h})^{-\beta}\}$ (again using Lemma 3.15 to bound the g_2 and g_3 terms)

$$\begin{aligned}
&h(h/\tilde{h})^{(m_2-1)/(m_2+1)} \left\langle \operatorname{Op}_{\tilde{h}}(g)u, u\right\rangle \\
&\quad \geqslant c\Gamma(h) x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)} \|u\|^2 \\
&\qquad + h(h/\tilde{h})^{(m_2-1)/(m_2+1)} (\left\langle \operatorname{Op}_{\tilde{h}}(g_2)u, u\right\rangle + \left\langle \operatorname{Op}_{\tilde{h}}(g_3)u, u\right\rangle) \\
&\quad \geqslant c(1-o(1)) h^{2+2/m_2-(1-\epsilon_0)/m_2-\epsilon_0/m_2(m_2+1)} \\
&\qquad \times \tilde{h}^{2+2/m_2-(m_2-1)/m_2-(3+\epsilon_0)/m_2+\epsilon_0/m_2(m_2+1)} \\
&\qquad \times x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)} \|u\|^2.
\end{aligned}$$

As in the previous subsection, we have $h/\varpi = o(1)$, so that $x_0 = o(1)$, and $h/\varpi \gg h^2$ so that (3.22) holds, which closes this part of the argument.

This concludes the study of the principal term in the commutator expansion. Of course we still have to control the lower order terms in the commutator expansion, which we do in the following Lemma. This is where it becomes very important that $\alpha = 1/(m_2+1) > 0$.

Lemma 3.16. *The symbol expansion of $[Q_1, a^w]$ in the h-Weyl calculus is of the form*

$$\begin{aligned}
[Q_1, a^w] = &\operatorname{Op}_h^w\Bigg(\left(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)(q_1(x,\xi)a(y,\eta) - q_1(y,\eta)a(x,\xi))|_{x=y,\xi=\eta} \\
&\quad + e(x,\xi) + r_3(x,\xi)\Bigg),
\end{aligned}$$

where e satisfies

$$\mathrm{Op}_h^w(e) \ll h\mathrm{Op}_h(\mathsf{H}(a)).$$

Proof. Since everything is in the Weyl calculus, only the odd terms in the exponential composition expansion are non-zero. Hence the h^2 term is zero in the Weyl expansion. Now according to Lemma 2.2 and the standard L^2 continuity theorem for h-pseudodifferential operators, we need to estimate a finite number of derivatives of the error:

$$|\partial^\gamma e_2| \leqslant Ch^3 \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R} \\ (y,\eta)\in T^*\mathbb{R}}} \sup_{|\rho|\leqslant M\ ,\rho\in\mathbb{N}^4} \left|\Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^3 q_1(x,\xi)a(y,\eta)\right|.$$

However, since $q_1(x,\xi) = \xi^2 + V_h(x)$, we have

$$D_x D_\xi q_1 = D_\xi^3 q_1 = 0,$$

so that

$$\begin{aligned}
\sigma(D)^3 & q_1(x,\xi)a(y,\eta)|_{x=y,\xi=\eta} \\
&= D_x^3 q_1 D_\eta^3 a|_{x=y,\xi=\eta} \\
&= -V_h'''(x)(\tilde{h}/h)^{2\beta}\Lambda_1'''((\tilde{h}/h)^\beta\eta) \\
&\quad \times \Lambda_2((\tilde{h}/h)^\alpha y)\chi(y)\chi(\eta) + r_3,
\end{aligned}$$

where r_3 is supported in $\{|(x,\xi)| \geqslant \delta_1\}$. Owing to the cutoffs $\chi(y)\chi(\eta)$ in the definition of a (and the corresponding implicit cutoffs in q_1), we only need to estimate this error in compact sets. The derivatives $h^\beta\partial_\eta$ and $h^\alpha\partial_y$ preserve the order of e_2 in h and increase the order in $\tilde{h}$, while the other derivatives lead to higher powers in $h/\tilde{h}$ in the symbol expansion. Hence we need only estimate e_2, as the derivatives satisfy similar estimates.

In order to estimate e_2, we again use conjugation to the 2-parameter calculus, and at some point invoke the 0-Gevrey assumption. We have

$$\|\mathrm{Op}_h^w(e_2)u\| = \|T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1}T_{h,\tilde{h}}u\| \leqslant \|T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1}\|_{L^2\to L^2}\|u\|,$$

by unitarity of $T_{h,\tilde{h}}$. But $T_{h,\tilde{h}}\mathrm{Op}_h^w(e_2)T_{h,\tilde{h}}^{-1} = \mathrm{Op}_{\tilde{h}}^w(e_2\circ\mathcal{B})$ and

$$\begin{aligned}
e_2\circ\mathcal{B} = &-h^3 V_h'''((h/\tilde{h})^\alpha X)(\tilde{h}/h)^{3\beta}\Lambda_1'''(\Xi) \\
&\times \Lambda_2(X)\chi(x)\chi(\xi) + r_3\circ\mathcal{B},
\end{aligned}$$

where r_3 is again microsupported away from the critical point (coming from the derivatives on $\chi(x)\chi(\xi)$. We recall that $V_h'''(x) = V'''(x) + W_h'''(x)$, where $W_h(x) = \Gamma(h)f((x-1)/x_0)$. As $f \in \mathcal{C}_c^\infty$, and we have already computed the value Γ, we know that

$$|W_h'''(x)| \leqslant C\Gamma x_0^{-3},$$

and hence

$$\begin{aligned}
|h^3(\tilde{h}/h)^{3\beta}\Lambda_1'''(\Xi)\Lambda_2(X)W_h'''((h/\tilde{h})^\alpha X)(\chi(x)\chi(\xi)| &\leqslant C\Gamma h^{3\alpha}\tilde{h}^{3\beta}x_0^{-3} \\
&= o(\Gamma(h)x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)}),
\end{aligned}$$

which is little-o of our computed lower bound on the quantization $h\mathrm{Op}_h(\mathsf{H}(a))$.

As for V, since $V' \in \mathcal{G}_\tau^0$, for x close to 1 satisfying (in the rescaled coordinates)

$$|X-1| \geqslant \left(\frac{h}{\tilde{h}}\right)^{-\alpha+\epsilon_1}, \ \ \epsilon_1 > 0,$$

$$\begin{aligned}|h^{3\alpha}\tilde{h}^{3\beta}V'''((h/\tilde{h})^{\alpha}X)| &\leqslant Ch^{3\alpha}\tilde{h}^{3\beta}\left|\left(\frac{h}{\tilde{h}}\right)^{\alpha}(X-1)\right|^{-2\tau}|V'((h/\tilde{h})^{\alpha}X)| \\ &\ll h^{2\alpha+\gamma}\tilde{h}^{3\beta}|V'((h/\tilde{h})^{\alpha}X)|\end{aligned}$$

provided $\epsilon_1 > 0$ is sufficiently small (as in the proof of Lemma 3.12). We also follow again the proof of Lemma 3.12 to break the analysis into several regions. The details are precisely the same.

On the other hand, we have for

$$|X-1| \leqslant \left(\frac{h}{\tilde{h}}\right)^{-\alpha+\epsilon_1}, \ \epsilon_1 > 0,$$

since $V_0'''(x) = \mathcal{O}(|x-1|^{\infty})$, then

$$|V_0'''((h/\tilde{h})^{\alpha}X)| = \mathcal{O}(h^{\infty}).$$

The error term must be estimated in terms of $h\mathsf{H}(a)$; we have shown that the error is always controlled by

$$\begin{aligned}&o(\Gamma(h)x_0^{-3-\epsilon_0+(2+\epsilon_0)/(m_2+1)}) \\ &\quad + o(h^{\alpha}\tilde{h}^{\beta})|V_{0,h}'((h/\tilde{h})^{\alpha}X)\left\langle \Xi\right\rangle^{-1-\epsilon_0}| + \mathcal{O}(h^{\infty}),\end{aligned}$$

which, when quantized, is controlled by little-o of our computed lower bound on the quantization $h\mathrm{Op}_h(\mathsf{H}(a))$.

□

The rest of the proof of Proposition 3.13 follows exactly as in the proof of Proposition 3.6.

Lastly, we show how to modify the preceding argument in the case of Proposition 3.14. The first step is to modify the function f and subsequently W_h and $V_{0,h}$. Since $V_0(x) \equiv 1$ on an interval $x-1 \in [-a,a]$, with $V_0'(x) < 0$ for $\pm(x-1) > a$, we choose the point $x_0 > 0$ to be the smallest number so that

$$-V_0'(x) \geqslant \frac{h}{\varpi(h)}, \ \ x_0 \leqslant x-1-a \leqslant \epsilon$$

and similarly for $-\epsilon \leqslant x-1+a \leqslant -x_0$. We choose also the same parameter $\varpi(h)$ as for Proposition 3.14. Again we can assume that $x_0 = o(1)$. Then choose $f \in \mathcal{C}_c^{\infty}(\mathbb{R}) \cap \mathcal{G}_{\tau}^0$ for some $\tau < \infty$, with $f(x) = -x^{2m_2+1}$ for $|x| \leqslant a + x_0$, and $f'(x) \leqslant 0$ for $x \geqslant 0$, $\operatorname{supp} f \subset [-a-2x_0, a+2x_0]$, satisfying

$$|\partial_x^k f| \leqslant C_k x_0^{-k}.$$

Following the proof of Proposition 3.8, we set the parameter $\widetilde{\Gamma}(h) = c_0\Gamma(h)$ for a small constant $c_0 > 0$ to be determined, and where $\Gamma(h)$ was computed in the course of the proof of Proposition 3.13. As in the proof of Proposition 3.8, we then take

$$W_h(x) = \widetilde{\Gamma}(h)f((x-1)),$$

and let

$$V_h(x) = V(x) + W_h(x)$$

We then follow the same arguments as in the proofs of Propositions 3.6 and 3.13, noting that the "smallness" assumption on the support of the microlocal cutoff φ in the x direction was to control lower order terms in Taylor expansions. As the function V_0 is constant and $f(x) = 1 - x^{2m_2+1}$ on $[-a,a]$, the smallness assumption

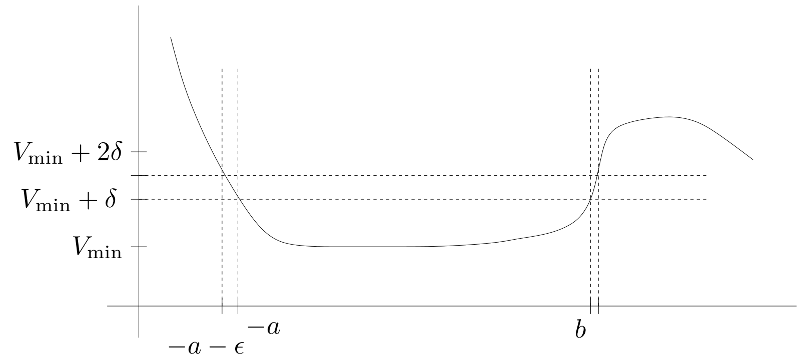

FIGURE 3. The potential V_0 near a local min, and the choice of parameters δ, $-a$, b, and ϵ.

translates into a small neighbourhood around $[1-a, 1+a]$. Hence in Proposition 3.14 we have assumed that $\operatorname{supp} \varphi \subset [1-a-\epsilon, 1+a+\epsilon]$. All of the error terms are treated similarly to the preceding proof. Then the same rescaling argument as in the proof of Proposition 3.8 proves Proposition 3.14.

$\square$

3.6. **Stable trapping and quasimodes.** Suppose $V_0(x)$ has an "honest" local minimum at $x = 0$ in the sense that V_0 is eventually increasing as one moves to the left or right of $x = 0$. That is, $V_0(0) = V_{\min}$, $V_0'(0) = 0$, $\pm V_0'(x) \geqslant 0$ for $\pm x \geqslant 0$ near 0, and if

$$\begin{cases} x_+ \geqslant 0 \\ x_- \leqslant 0 \end{cases}$$

are the smallest positive/negative values with $V_0'(x) \neq 0$ for $\pm x > \pm x_\pm$, then $\pm V'(x) > 0$ for $\pm x > \pm x_\pm$ in some neighbourhood. This means there exists $\delta > 0$ such that for each $y \in [V_{\min}, V_{\min} + 2\delta]$, the sets $\{x : V_0(x) \leqslant y\}$ have a non-empty compact connected component containing $x = 0$. By shrinking $\delta > 0$ if necessary, we may also assume that $V_0(x)$ is a convex function on the connected component containing $x = 0$.

Let $-a < 0$ be the largest negative number such that $V_0(-a) = V_{\min} + \delta$, and let $b > 0$ be the smallest positive number such that $V_0(b) = V_{\min} + \delta$. By again shrinking $\delta > 0$ if necessary, we may assume that $V_0'(-a) < 0$ and $V_0'(b) > 0$. Choose also $\epsilon > 0$ such that $V_0(-a-\epsilon) \geqslant V_{\min} + 3\delta/2$ and $V_0(b+\epsilon) \geqslant V_{\min} + 3\delta/2$ (again shrinking $\delta > 0$ if necessary). Figure 3 is a picture of the setup.

Let

$$\widetilde{V}(x) = \begin{cases} V(x) = V_0(x) + h^2 V_1(x), x \in [-a-\epsilon, b+\epsilon], \\ \beta x^2, |x| \gg 1, \end{cases}$$

where $\beta > 0$ is an appropriate constant so that $\widetilde{V}$ can be assumed convex. In particular, we may assume that $\widetilde{V}^{-1}(E) \subset [-a, b]$ for $E \in [V_{\min} + \delta/2, V_{\min} + 2\delta/3]$, by taking $h > 0$ sufficiently small.

Let $L = (hD_x)^2 + \widetilde{V}(x)$. For $h > 0$ sufficiently small, Weyl's law implies there exists $\sim h^{-1}$ eigenvalues E of the operator L in the interval $E \in [V_{\min} + \delta/2, V_{\min} +$

$2\delta/3]$. Fix such an eigenvalue, and let $\varphi(x)$ be the (normalized) associated eigenfunction. Then, since $E \in [V_{\min} + \delta/2, V_{\min} + 2\delta/3]$, we have in particular that

$$\varphi(x) = \mathcal{O}(h^\infty), \;\; x \leqslant -a, \, x \geqslant b.$$

Let $\chi(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ be a smooth function such that $\chi(x) \equiv 1$ on $[-a, b]$ with support in $[-\epsilon - a, b + \epsilon]$. Thus,

$$\chi(x)\varphi(x) = \varphi(x) + \mathcal{O}(h^\infty),$$

and

$$\begin{aligned} L\chi\varphi &= \chi L\varphi + [L, \chi]\varphi \\ &= E\chi\varphi + \mathcal{O}(h^\infty), \end{aligned}$$

since $[L, \chi]$ is supported on the set where $\varphi = \mathcal{O}(h^\infty)$.

But since $\widetilde{V} = V$ on the set $[-a - \epsilon, b + \epsilon]$, we also have

$$((hD_x)^2 + V(x))\chi\varphi = E\chi\varphi + \mathcal{O}(h^\infty).$$

As $\|\chi\varphi\| = 1 - \mathcal{O}(h^\infty)$, we have

$$\|((hD_x)^2 + V(x) - E)\chi\varphi\| = \mathcal{O}(h^\infty)\|\chi\varphi\|,$$

and so evidently for any N, there exists C_N such that for any compactly supported function $\tilde{\chi}$ such that $\tilde{\chi} = 1$ on $\operatorname{supp}\chi$,

$$\|\tilde{\chi}((hD_x)^2 + V(x) - E)^{-1}\tilde{\chi}\chi\varphi\| \geqslant C_N h^{-N}\|\chi\varphi\|. \tag{3.23}$$

Remark 3.17. Of course, if we know more about the structure of the function $V_0(x)$ near an honest local minimum, then we can say more. For example, we can construct WKB approximations as quasimodes, and then stationary phase can tell us much more detailed information about the quasimodes. However indirect, the Weyl law method presented here is in a sense more robust, and does not really require intimate knowledge of $V_0(x)$ near the minimum.

4. Proof of Theorem 1

We are now able to prove Theorem 1. Since we have assumed that the trapped set has only finitely many connected components, this implies that the function $A(x)$ has only finitely many critical values, which consequently occur in a compact set. Let $A_1, A_2, \ldots A_k$ be the critical values, and let $K_l = \{A(x) = A_l\}$ be the critical sets. There are two cases to consider.

Case 1: The function $V_0(x) = A^{-2}(x)$ has a local minimum. Then apply (3.23) to conclude there are highly localized quasimodes, and the resolvent therefore blows up faster than any polynomial (at least along a subsequence).

Case 2: The function $V_0(x) = A^{-2}(x)$ has no minima. In this case, each critical value of the function $A^{-2}(x)$ is of either unstable or transmission inflection type (whether infinitely degenerate or not). The important thing to observe is that there are only finitely many critical values, and they are all isolated in the two-dimensional phase space in the following sense: if K_l is disconnected for some l, then assume K_l has only two connected components (the finite case being similar). The connected components of K_l are separated by a maximum, say K_j (or minimum, but this would be Case 1). The stable and unstable manifolds associated to the flow around K_j form a *global* separatrix, separating the complete flowouts of the two components of K_l. This allows us to microlocalize and glue together the trapping estimates

examined in detail in Subsections 3.1-3.5 (see Appendix A, [Chr08], [CM13], and [DV12]). The rest of the proof proceeds precisely as in the proof of [CW11, Theorem 2].

5. Proof of Corollary 1.5

In this section, we recall the functional theoretic argument which connects resolvent estimates to local smoothing estimates, and in the process prove Corollary 1.5. The technique is often called a "TT^*" argument, however we have already used T in our time interval, so instead we use an AA^* argument.

Let $u_0 \in \mathcal{S}(X)$ and $\chi \in \mathcal{C}_c^\infty(X)$ and let A be the operator

$$Au_0 = \chi e^{it\Delta} u_0,$$

acting on $L^2(X)$. We want to show

$$A : L^2(X) \to L^2([0,T]; H^s(X))$$

for some $s > 0$ is bounded. By duality, this is equivalent to the adjoint A^* being bounded

$$A^* : L^2([0,T]; H^{-s}(X)) \to L^2(X),$$

which is equivalent to the boundedness of the composition

$$AA^* : L^2([0,T]; H^{-s}(X)) \to L^2([0,T]; H^s(X)).$$

Computing directly, we get

$$AA^* f(t) = \int_0^T \chi e^{i(t-\tau)\Delta} \chi f(\tau) d\tau.$$

Now let u be defined by

$$u(x,t) = \int_0^T e^{i(t-\tau)\Delta} \chi f(\tau) d\tau.$$

Since we are only interested in the time interval $[0,T]$, we extend f to be 0 for $t \notin [0,T]$. We write

$$\begin{aligned} AA^* f(t) &= \int_0^t \chi e^{i(t-\tau)\Delta} \chi f(\tau) d\tau + \int_t^T \chi e^{i(t-\tau)\Delta} \chi f(\tau) d\tau \\ &=: \chi u_1(t) + \chi u_2(t), \end{aligned}$$

and calculate

$$(D_t - \Delta) u_j = (-1)^j i \chi f. \tag{5.1}$$

Thus boundedness of AA^* will follow if we prove u satisfying (5.1) satisfies

$$\|\chi u\|_{L^2([0,T];H^s)} \leqslant \|f\|_{L^2([0,T];H^{-s})}.$$

Replacing $\pm if$ with f in equation (5.1) and taking the Fourier transform in time, $t \mapsto z$, results in the following equation for $\hat{u}$ and $\hat{f}$:

$$(z - \Delta)\hat{u}(z, \cdot) = \chi \hat{f}(z, \cdot). \tag{5.2}$$

Since $f(t, \cdot)$ is supported only in $[0,T]$, $\hat{f}(z, \cdot)$ and $\hat{u}(z, \cdot)$ are holomorphic, bounded, and satisfy (5.2) in $\{\operatorname{Im} z < 0\}$. Let $z = \tau - i\eta$, $\eta > 0$ sufficiently small. Since the Fourier transform is an $L^2 H$ isometry for any Hilbert space H, we want to estimate

$$\|\chi \hat{u}(z, \cdot)\|_{H^s(X)} \leqslant C \|\hat{f}(z, \cdot)\|_{H^{-s}(X)}$$

uniformly in z.

For this, we observe that if we know

$$\|\chi(-\Delta+z)^{-1}\chi\|_{L^2\to L^2} \leqslant C|z|^{-r}$$

for some $r \geqslant 0$ and $\operatorname{Im} z = -\eta < 0$ fixed, then a standard interpolation argument gives

$$\|\chi(-\Delta+z)^{-1}\chi\|_{L^2\to H^2} \leqslant C|z|^{1-r},$$

and hence

$$\|\chi(-\Delta+z)^{-1}\chi\|_{L^2\to H^{2r}} \leqslant C.$$

Interpolating again, we get

$$\|\chi(-\Delta+z)^{-1}\chi\|_{H^{-r}\to H^r} \leqslant C.$$

Thus

$$\begin{aligned}
\|\chi u\|_{L^2([0,T];H^r(X))} &\leqslant e^{\eta T}\|e^{-\eta t}\chi u(t)\|_{L^2([0,T];H^r(X))} \\
&\leqslant Ce^{\eta T}\|\chi\hat{u}(\tau - i\eta)\|_{L^2(\mathbb{R};H^r(X))} \\
&\leqslant Ce^{\eta T}\|\hat{f}(\tau - i\eta)\|_{L^2(\mathbb{R};H^{-r}(X))} \\
&\leqslant Ce^{\eta T}\|e^{-\eta t}f(t)\|_{L^2([0,T];H^{-r}(X))} \\
&\leqslant Ce^{\eta T}\|f(t)\|_{L^2([0,T];H^{-r}(X))}.
\end{aligned}$$

Hence

$$\int_0^T \|\chi u\|^2_{H^r(X)} dt \leqslant Ce^{\eta T}\int_0^T \|f\|^2_{H^{-r}(X)} dt,$$

or AA^* is bounded.

We remark in passing that this argument works for any $r \geqslant 0$, along a strip where $\operatorname{Im} z = -\eta < 0$. If we examine the case where $\|\chi(-\Delta+z)^{-1}\chi\|_{L^2\to L^2}$ blows up as $\operatorname{Im} z \to 0$ (as in Case 2 of Theorem 1), we can use instead the trivial bound

$$\|\chi(z-\Delta)\chi\|_{L^2\to L^2} \leqslant \frac{1}{|\operatorname{Im} z|} = \frac{1}{\eta}$$

in this case to get a zero derivative smoothing effect. But of course we already knew such an estimate must be true (even without spatial cutoffs) from the $L^2(X)$ conservation law. The point is that the blowup of the resolvent is perfectly consistent with our physical intuition in this problem.

6. An Application: Spreading of Quasimodes for some Partially Rectangular Billiards

In this section, we apply the microlocal estimates proved in the previous sections to prove a spreading result for rather weak quasimodes for the Laplacian in partially rectangular billiards. The main result is that if a partially rectangular billiard opens "outward" in at least one wing, then any $\mathcal{O}(\lambda^{-\epsilon})$ quasimode must spread to outside of any $\mathcal{O}(\lambda^{-\epsilon})$ neighbourhood of the rectangular part. This result holds for any $\epsilon > 0$.

These results are similar in spirit to results of Burq-Zworski [BZ04] and of Burq-Hassell-Wunsch [BHW07], but the techniques of proof are different. Let us be precise.

Let $\Omega \subset \mathbb{R}^2$ be a planar domain with boundary in the 0-Gevrey class $\mathcal{G}^0_\tau$ for $\tau < \infty$, and let $R = [-a,a]\times[-\pi,\pi] \subset \mathbb{R}^2$ be a rectangle with boundary consisting of the

two sets of parallel segments $\partial R = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 = [-a,a]\times\{\pi\}\cup[-a,a]\times\{-\pi\}$. Assume $R \subset \Omega$ and $\Gamma_1 \subset \partial\Omega$ but $\mathring{\Gamma}_2 \cap \partial\Omega = \emptyset$. We assume that for (x,y) in a neighbourhood of R, $\partial\Omega$ is symmetric about the line $y=0$. Let $Y(x) = \pi + r(x)$ be a graph parametrization of the boundary curve $\partial\Omega$ for (x,y) near $[-a,a]\times\{\pi\}$.

Theorem 3. *Consider the quasimode problem on Ω:*

$$\begin{cases} (-\Delta - \lambda^2)u = E(\lambda)\|u\|_{L^2}, \ \text{on } \Omega, \\ Bu = 0, \ \text{on } \partial\Omega, \end{cases}$$

where $B = I$ or $B = \partial_\nu$ (either Dirichlet or Neumann boundary conditions).

Assume that $\pm r'(x) > 0$ for at least one of $\pm(x \mp a) > 0$ (that is, the boundary curves "outward" away from the rectangular part of the boundary for at least one side). Fix $\epsilon > 0$. If $E(\lambda) = \mathcal{O}(\lambda^{-\epsilon})$ as $\lambda \to \infty$ and $\mathrm{WF}_{\lambda^{-1}} u$ vanishes outside a neighbourhood of size $\mathcal{O}(\lambda^{-\epsilon})$ of R, then $u = \mathcal{O}(\lambda^{-\infty})$ on Ω.

See Figures 4 and 5 for examples where the theorem applies.

Remark 6.1. It should be clear from the proof that the 0-Gevrey assumption need only hold in a neighbourhood of the rectangular region R. It should also be clear from the proof that the symmetry in y is not necessary; it is enough that the vertical distance is increasing on at least one side of the rectangular part (see Figure 6).

It is believed that there can be a sequence of eigenfunctions which concentrate on the entire rectangular part. This result does not preclude this, as the neighbourhood in which the theorem applies shrinks to the rectangular part as $\lambda \to \infty$. However, it gives a lower bound on how fast a quasimode may concentrate on the rectangular part under the assumptions of the Theorem. Moreover, the proof is meant to be extremely elementary given the estimates established in the first part of this paper. It is possible that in some special cases, with a little more care, the assumptions can be weakened to include any quasimode localized in a sufficiently small neighbourhood independent of λ.

We also remark that this theorem does not apply to the famous Bunimovich stadium, since the boundary in that case is neither 0-Gevrey smooth, nor does it open outward on either side of the rectangular part. Indeed, following the first part of the proof of Theorem 3 below, the effective potential curves "upward" as $y \sim -(\pi - (x \mp a)^2)^{1/2}$, where 2π is the height of the Bunimovich stadium and $2a$ is the width of the rectangular part. A very sketchy heuristic is that the lowest energy quasimode sitting in this potential well occurs when the potential well is approximately λ^{-1} deep. That is, it should be concentrated in the set where

$$y + \pi = \pi - (\pi - (x \mp a)^2)^{1/2} \sim \lambda^{-1},$$

which occurs when $x \mp a \sim \lambda^{-1/2}$, or within a $\lambda^{-1/2}$ neighbourhood of the rectangular part. This is precisely the type of behaviour that Theorem 3 rules out.

Proof. The first step in the proof is to straighten the boundary near the rectangular part and then approximately separate variables. The boundary Γ near R is given by $y = \pm Y(x) = \pm(\pi + r(x))$ for $x \in [-a-\delta, a+\delta]$ for some $\delta > 0$. Write $P_0 = -\partial_x^2 - \partial_y^2$ for the flat Laplacian. We will "straighten the boundary" near R and compute the corresponding change in the metric. From this we will get a non-flat Laplace-Beltrami operator which is almost separable. Recall u solves

$$P_0 u = \lambda^2 u + E(\lambda)\|u\|$$

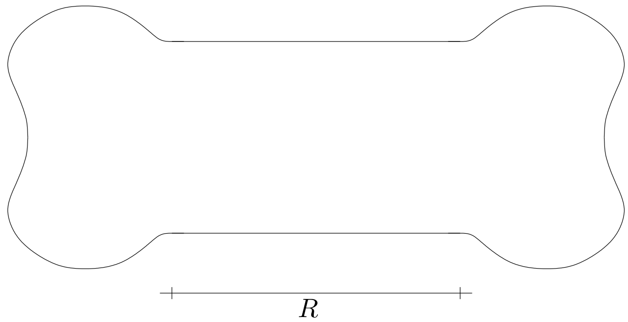

Figure 4. A partially rectangular billiard opening "outward" away from the rectangular part. No $\mathcal{O}(\lambda^{-\epsilon})$ quasimode can concentrate in an $\mathcal{O}(\lambda^{-\epsilon})$ neighbourhood of R.

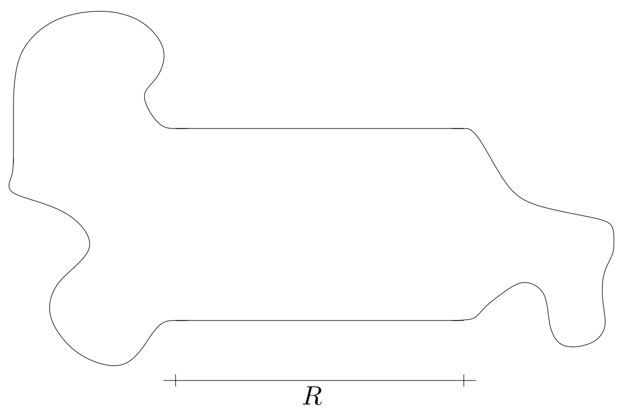

Figure 5. A partially rectangular billiard opening "outward" away from the rectangular part on one side and "inward" on the other. The same conclusion applies.

and if $\chi(x,\lambda)$ satisfies $\chi \equiv 1$ on $\{|x| \leqslant a+\lambda^{-\epsilon}\}$, with support in, say $\{|x| \leqslant a+2\lambda^{-\epsilon}\}$ then

$$\chi u = u + \mathcal{O}(\lambda^{-\infty})\|u\|$$

in any Sobolev space. Let $\tilde{R} = \{(x,y) \in \Omega : |x| \leqslant a + 4\lambda^{-\epsilon}\}$ be a shrinking neighbourhood of R in Ω, which is slightly larger than the set where χu is supported. We change variables $(x,y) \mapsto (x',y')$ in $\tilde{R}$ in a way which straightens out the boundary:

$$\begin{cases} x = x', \\ y = y'Y(x'). \end{cases}$$

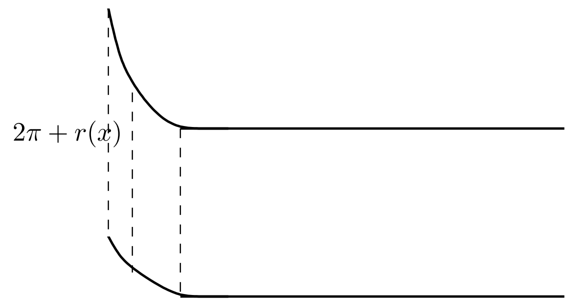

FIGURE 6. Part of a partially rectangular billiard where the local symmetry in y does not hold, but the vertical distance on one side is an increasing function away from the rectangular part. The same conclusion applies in this case as well by trivially modifying the proof.

Thus when $y = \pm Y(x) = \pm Y(x')$, $y' = \pm 1$. We have

$$\begin{aligned} g &= dx^2 + dy^2 \\ &= (dx')^2 + (Y dy' + y'Y'(x')dx')^2 \\ &= (1 + A)(dx')^2 + 2B dx' dy' + Y^2 (dy')^2, \end{aligned}$$

where

$$A = (y'Y'(x'))^2,$$

and

$$B = y'Y'Y.$$

In matrix notation,

$$g = \begin{pmatrix} 1+A & B \\ B & Y^2 \end{pmatrix}.$$

Let us drop the cumbersome (x', y') notation and write (x, y) instead. In order to compute Δ_g in these coordinates, we need $|g|$ and g^{-1}. We compute

$$\begin{aligned} |g| &= Y^2(1 + A) - B^2 \\ &= Y^2 + y^2 Y^2 (Y')^2 - y^2 Y^2 (Y')^2 \\ &= Y^2. \end{aligned}$$

Hence

$$g^{-1} = Y^{-2} \begin{pmatrix} Y^2 & -B \\ -B & 1+A \end{pmatrix}.$$

For our quasimode u as above, we have after a tedious computation

$$\begin{aligned} -\Delta_g u = -\Big(& \partial_x^2 + Y^{-2}(1+A)\partial_y^2 + Y'Y^{-1}\partial_x - 2BY^{-2}\partial_x\partial_y \\ & - Y^{-1}(B/Y)_x \partial_y - Y^{-1}(B/Y)_y \partial_x + Y^{-1}((1+A)/Y)_y \partial_y \Big) u. \end{aligned}$$

Now let $\tilde{\chi}(x, \lambda)$ be a smooth function such that $\tilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$ with support in $\{|x| \leqslant a + 4\lambda^{-\epsilon}\}$. Let us also assume for simplicity that we have normalized

$\|u\| = 1$. Then, since Δ_g does not enlarge the wavefront set, we have

$$\begin{aligned} -\Delta_g u &= -\tilde{\chi}\Delta_g u + \mathcal{O}(\lambda^{-\infty}) \\ &= -\Delta_g \tilde{\chi} u - [\tilde{\chi}, \Delta_g]u + \mathcal{O}(\lambda^{-\infty}) \\ &= -\Delta_g \tilde{\chi} u + \mathcal{O}(\lambda^{-\infty}) \end{aligned}$$

by the support properties of $\tilde{\chi}$, χ, and the wavefront assumption on u. Hence we have

$$-\Delta_g \tilde{\chi} u = \lambda^2 \tilde{\chi} u + E(\lambda)\|\tilde{\chi} u\| + \mathcal{O}(\lambda^{-\infty}).$$

Observe now that the functions A and B are both $\mathcal{O}(Y'(x)) = \mathcal{O}(r'(x))$, which for $(x, y) \in \tilde{R}$ is $\mathcal{O}(\lambda^{-\infty})$. Hence,

$$-\Delta_g \tilde{\chi} u = P_2 \tilde{\chi} u + \mathcal{O}(\lambda^{-\infty}),$$

where $P_2 = -\partial_x^2 - Y^{-2}(x)\partial_y^2$. That is,

$$P_2 \tilde{\chi} u = \lambda^2 \tilde{\chi} u + E(\lambda)\|\tilde{\chi} u\| + \mathcal{O}(\lambda^{-\infty}).$$

Since $\tilde{\chi} u$ is supported in $\tilde{R}$, which in these coordinates is just the rectangle

$$\tilde{R} = [-a - 4\lambda^{-\epsilon}, a + 4\lambda^{-\epsilon}]_x \times [-1, 1]_y,$$

we can expand in a Fourier basis (with appropriate boundary conditions):

$$\tilde{\chi} u = \sum_{k \in \mathbb{Z}} \tilde{\chi} u_k(x) e_k(y).$$

Let $\beta_k^2 \sim k^2$ be the eigenvalues in the y direction, so that $-\partial_y^2 e_k = \beta_k^2 e_k$, and let $P_k = -\partial_x^2 + \beta_k^2 Y^{-2}(x)$, so that

$$P_2 \tilde{\chi} u = \sum_{k \in \mathbb{Z}} e_k(y) P_k \tilde{\chi} u_k.$$

Now we rescale $h = \beta_k^{-1}$ and write

$$P(h) = -h^2 \partial_x^2 + Y^{-2}(x),$$

so that $w = \tilde{\chi} u_k$ must satisfy a semiclassical equation of the form

$$P(h) w = z w + \widetilde{E}\|w\| + \mathcal{O}(\lambda^{-\infty}),$$

where $z = h^2\lambda^2$ and $\widetilde{E} = h^2 E(\lambda)$.

Now on $\tilde{R}$, the function Y^{-2} satisfies $\pi^{-2} - \delta/2 \leqslant Y^{-2} \leqslant \pi^{-2}$ for some $\delta > 0$ independent of $\lambda \to \infty$. If $z < \pi^{-2} - \delta$, say, then $P(h) - z$ is elliptic and satisfies

$$\|(P(h) - z)^{-1} \tilde{\chi}\|_{L^2 \to L^2} \leqslant C.$$

Hence for z in this range, we have

$$\|w\| = C\widetilde{E}\|w\| + \mathcal{O}(\lambda^{-\infty}),$$

which implies $\|w\| = \mathcal{O}(\lambda^{-\infty})$, since in this range of z, we have $h^2 \leqslant C\lambda^{-2}$, which implies

$$\widetilde{E} = h^2 E(\lambda) = \mathcal{O}(\lambda^{-2})$$

in any case.

Now if $z \geqslant \pi^{-2} + \delta$, then $\{\xi^2 + Y^{-2}(x) = z\}$ has no critical points, so for these values of z, $P(h) - z$ obeys a non-trapping estimate:

$$\|(P(h) - z)^{-1} \tilde{\chi}\|_{L^2 \to L^2} \leqslant C h^{-1}.$$

Applying the same argument as in the previous case and observing that

$$h^{-1}\widetilde{E} = hE = \mathcal{O}(\lambda^{-\epsilon})$$

yields again $w = \mathcal{O}(\lambda^{-\infty})$.

For the remaining range of z, we must use our estimates from Propositions 3.8 and 3.14. We have assumed that $E(\lambda) = \mathcal{O}(\lambda^{-\epsilon}) = \mathcal{O}(h^{\epsilon})$ for some $\epsilon > 0$ fixed.

Since we are now in the region where

$$\pi^{-2} - \delta \leqslant z \leqslant \pi^{-2} + \delta,$$

we have $h \sim \lambda^{-1}$, so $E(\lambda) = \mathcal{O}(\lambda^{-\epsilon}) \sim \mathcal{O}(h^{\epsilon})$ as $h \to 0$, and $\mathcal{O}(h^{\infty})$ is equivalent to $\mathcal{O}(\lambda^{-\infty})$. We can further microlocalize and apply Proposition 3.8 or 3.14 (as the case may be) with $\eta \ll \epsilon$ to get

$$\|w\| \leqslant C\lambda^{\eta} E(\lambda)\|w\| + \mathcal{O}(\lambda^{-\infty}),$$

which again implies $\|w\| = \mathcal{O}(\lambda^{-\infty})$.

□

Appendix A. A User's Guide to Resolvent Gluing

Recently, a number of authors have used various constructions to glue together resolvent estimates from different situations (see, for example, [Chr08,CW11,DV12, CM13] and the present work). For example, the best estimates near classically trapped sets are typically very local (or microlocal) in nature, but in most reasonable situations, the geodesic flow tends to infinity uniformly outside a small neighbourhood of the trapping. Hence one expects the behaviour at spatial infinity to act somewhat independently of the behaviour near the trapped set. As a result, one tries to "glue" the microlocal estimates near the trapping into the non-trapping estimates at infinity. In this appendix, we present a simple gluing technique which works in the cases of interest in this paper; namely in one dimensional semiclassical potential scattering.

Let $P = -h^2\partial^2 + V(x)$ be a semiclassical Schrödinger operator in one spatial dimension. We assume the potential $V(x)$ is a short range perturbation of the inverse square potential (with one or two "ends"). That is, we assume that for some $R > 0$,

$$|x| \geqslant R \implies |\partial^k(V(x) - x^{-2})| \leqslant C_k \langle x \rangle^{-2-k}.$$

Let $p = \xi^2 + V(x)$ be the symbol of P. In this case, it is well known that any critical points of H_p are contained in a compact set, and there exists a large, positive number M and a symbol $\tilde{p}$ which is globally non-trapping, and $p = \tilde{p}$ for $x \geqslant M - 1$. Then there are a wide selection of non-trapping estimates available for $\widetilde{P} = \mathrm{Op}_h(\tilde{p})$ with a $\mathcal{O}(h^{-1})$ bound on the cutoff resolvent. As we have seen in this paper, if there are any stable critical elements of H_p, then there are many nearby trajectories which do not escape to infinity, and no gluing techniques are necessary, since we already know the resolvent blows up rapidly. Hence this construction will only apply when all the critical elements of H_p are at least weakly unstable.

Increasing M later on if necessary, we first select a number of cutoffs. Let $\tilde{\chi} \in \mathcal{C}_c^{\infty}(\mathbb{R})$, $0 \leqslant \tilde{\chi} \leqslant 1$, $\tilde{\chi} \equiv 1$ for $|x| \leqslant 2M$. Let $\tilde{\chi}_2 \in \mathcal{C}_c^{\infty}(\mathbb{R})$ be equal to 1 on $\operatorname{supp}\tilde{\chi}$ with slightly larger support. Let $\rho_s \in \mathcal{C}^{\infty}(\mathbb{R})$ be a smooth function, $\rho_s > 0$, $\rho_s(x) \equiv 1$ on $\operatorname{supp}\tilde{\chi}_2$, $\rho(x) = \langle x \rangle^s$ for very large $|x|$. Let $\Gamma \in \mathcal{C}_c^{\infty}(\mathbb{R})$ be a cutoff equal to 1 on $\{|x| \leqslant M - 1\}$ with support in $\{|x| \leqslant M\}$. That means that the non-trapping symbol $\tilde{p}$ equals p on the support of $1 - \Gamma$. The idea is that things

are well-behaved on the support of $1-\Gamma$, and on the support of Γ, we decompose Γ into a sum of cutoffs where we have "black-box" microlocal estimates established through other means. The most important tool for gluing all of these estimates together is the propagation of singularities lemma in the next subsection.

A.1. **Propagation of Singularities.** The rough idea behind propagation of singularities is that if two regions in phase space are connected by the H_p flow, then the L^2 mass in one region is controlled by the L^2 mass in the other, modulo a term involving P. The very nature of requiring the H_p flow to move from one region to another means these estimates are inherently non-trapping.

In order to motivate the more general statement below, let us describe a baby version of propagation of singularities in the very special case that the operator is $P = hD_x$. Of course, it is well known that if $H_p \neq 0$, then P is microlocally conjugate to hD_x, so this is not as ridiculous as it initially seems. Indeed, the rough heuristic we sketch here can be fixed up to provide an alternative proof to Lemma A.1 below. The proof of this Lemma in [Chr08] proceeds by the traditional method of the original proof of Hörmander (see the original in [Hör71] or the presentation in [Tay81]).

Consider a function $u(x)$ of one variable. Let $a < b$ be two points in $\mathbb{R}$. We show that the L^2 mass of u in a neighbourhood of size 1 about a is controlled by the mass of u near b in a neighbourhood of size K modulo a term involving P. Perhaps more importantly, the constant on the term near b is comparable to $K^{-1/2}$. That is, by enlarging the control region to size K, we can make the constants in our estimates small.

For $s, t \geqslant 0$, we write

$$u(a+s) - u(b+t) = \int_{a+s}^{b+t} u'(r) dr = \frac{i}{h} \int_{a+s}^{b+t} P u dr.$$

Rearranging and taking the absolute value squared and applying Hölder's inequality to the integral, we get

$$\begin{aligned} |u(a+s)|^2 &\leqslant 2|u(b+t)|^2 + 2h^{-2} \left(\int_{a+s}^{b+t} |Pu| dr \right)^2 \\ &\leqslant 2|u(b+t)|^2 + 2h^{-2}((b+t)-(a+s)) \|Pu\|^2_{L^2(a+s,b+t)}. \end{aligned}$$

We now integrate in $0 \leqslant s \leqslant 1$:

$$\|u\|^2_{L^2(a,a+1)} \leqslant 2|u(b+t)|^2 + 2h^{-2}((b+t)-a) \|Pu\|^2_{L^2(a,b+t)}.$$

We follow this by integrating in $0 \leqslant t \leqslant K$ to get

$$K\|u\|^2_{L^2(a,a+1)} \leqslant 2\|u\|^2_{L^2(b,b+K)} + 2Kh^{-2}(b+K-a)\|Pu\|^2_{L^2(a,b+K)}.$$

We conclude that

$$\|u\|_{L^2(a,a+1)} \leqslant \frac{\sqrt{2}}{\sqrt{K}} \|u\|_{L^2(b,b+K)} + \sqrt{2} h^{-1} (b+K-a)^{1/2} \|Pu\|_{L^2(a,b+K)}.$$

The more general version of this idea is given in the following Lemma from [Chr08].

Lemma A.1. *Let $\widetilde{V}_1, \widetilde{V}_2 \Subset M$, and for $j = 1, 2$ let $V_j \Subset T^*M$,*

$$V_j := \{(x,\xi) \in T^*M : x \in \widetilde{V}_j, \ |p(x,\xi) - E| \leqslant \alpha\},$$

for some $\alpha > 0$. Suppose the $\widetilde{V}_j$ satisfy $\mathrm{dist}_g(\widetilde{V}_1, \widetilde{V}_2) = L$, and assume

$$\left\{ \begin{array}{l} \exists C_1, C_2 > 0 \text{ such that } \forall \rho \text{ in a neighbourhood of } V_1, \\ \exp(tH_p)(\rho) \in V_2 \text{ for} \\ \sqrt{E}(L + C_1) \leqslant t \leqslant \sqrt{E}(L + C_1 + C_2). \end{array} \right. \tag{A.1}$$

Suppose $A \in \Psi_h^{0,0}$ is microlocally equal to 1 in V_2. If $B \in \Psi_h^{0,0}$ and $\mathrm{WF}(B) \subset V_1$, then there exists a constant $C > 0$ depending only on C_1, C_2 such that

$$\begin{aligned} \|Bu\| \quad \leqslant \quad & CLh^{-1}\|B\|_{\mathcal{H}\to\mathcal{H}} \|(P(h) - z)u\| + 2(E+\alpha)^{3/4} \frac{(C_1+1)}{\sqrt{C_2}} \|B\|_{\mathcal{H}\to\mathcal{H}} \|Au\| \\ & + \mathcal{O}(h)\|\widetilde{B}u\|, \end{aligned}$$

where

$$\widetilde{B} \equiv 1 \text{ on } \cup_{0 \leqslant t \leqslant \sqrt{E}(L+C_1+C_2)} \exp(tH_p)(\mathrm{WF}B).$$

A.2. **The gluing.** Now fix an energy level z. As described briefly above, we now assume that the function Γ can be further decomposed as a sum of pseudodifferential cutoffs:

$$\Gamma = \sum_{j=1}^{N} \Gamma_j,$$

where for each Γ_j, $\Gamma_j = 1$ on a set where $H_p = 0$, and we have a microlocal black box estimate of the form

$$\|\Gamma_j u\| \leqslant \frac{\alpha_j(h)}{h} \|(P - z)\Gamma_j u\|.$$

Here, it is necessary that $\alpha_j(h) = \mathcal{O}(h^{-K})$ for the technique to work (this is the same condition in, for example, [DV12]). However, in this paper, we have seen that for the present applications, this estimate is true with $K = 1 + \epsilon$. We will assume this is true to save a little bit of work later (but we will point out where the extra step would be needed). We also require that each $\Gamma_j \in \mathcal{S}^0$. Naturally, at least one of the Γ_js will be supported for large frequencies ξ where $P - z$ is elliptic, so in this case the corresponding $\alpha_j(h) = \mathcal{O}(h)$.

We write for $s < -1/2$ and some positive numbers c_1, c_2:

$$\begin{aligned} \|\rho_{-s}(P - z)u\|^2 &\geqslant c_1(\|\rho_{-s}(1 - \Gamma)(P - z)u\|^2 + \|\Gamma(P - z)u\|^2) \\ &\geqslant c_2(\|\rho_{-s}(1 - \Gamma)(P - z)u\|^2 + \sum_{j=1}^{N} \|\Gamma_j(P - z)u\|^2). \end{aligned}$$

There is no ρ_{-s} in the terms coming from Γ because ρ_{-s} was assumed to equal 1 on $\operatorname{supp}\Gamma$. We now write

$$\begin{aligned} \|\rho_{-s}(1 - \Gamma)(P - z)u\|^2 &= \|\rho_{-s}(P - z)(1 - \Gamma)u + [P, \Gamma]u\|^2 \\ &= \|\rho_{-s}(P - z)(1 - \Gamma)u\|^2 + \|[P, \Gamma]u\|^2 \\ &\quad + 2\,\mathrm{Re}\, \langle \rho_{-s}(P - z)(1 - \Gamma)u, [P, \Gamma]u \rangle \\ &\geqslant \|\rho_{-s}(P - z)(1 - \Gamma)u\|^2 + \|[P, \Gamma]u\|^2 \\ &\quad - 2\|\rho_{-s}(P - z)(1 - \Gamma)u\| \|[P, \Gamma]u\|. \end{aligned}$$

Expanding similarly the terms involving the Γ_js, we get

$$\begin{aligned}\|\rho_{-s}(P-z)u\|^2 \geqslant \|\rho_{-s}(P-z)(1-\Gamma)u\|^2 + \|[P,\Gamma]u\|^2 \\ -2\|\rho_{-s}(P-z)(1-\Gamma)u\|\|[P,\Gamma]u\| \\ + \sum_{j=1}^{N} \big(\|\rho_{-s}(P-z)\Gamma_j u\|^2 + \|[P,\Gamma_j]u\|^2 \\ -2\|\rho_{-s}(P-z)\Gamma_j u\|\|[P,\Gamma_j]u\|\big).\end{aligned}$$

Applying Cauchy's inequality yields for $\eta > 0$:

$$\begin{aligned}&\|\rho_{-s}(P-z)u\|^2 \\ &\geqslant \|\rho_{-s}(P-z)(1-\Gamma)u\|^2 + \|[P,\Gamma]u\|^2 \\ &\quad -2\left(\eta\|\rho_{-s}(P-z)(1-\Gamma)u\|^2 + 4\eta^{-1}\|[P,\Gamma]u\|^2\right) \\ &\quad + \sum_{j=1}^{N} \big(\|\rho_{-s}(P-z)\Gamma_j u\|^2 + \|[P,\Gamma_j]u\|^2 \\ &\quad -2(\eta\|\rho_{-s}(P-z)\Gamma_j u\|^2 + 4\eta^{-1}\|[P,\Gamma_j]u\|^2)\big),\end{aligned}$$

which, by taking $\eta > 0$ sufficiently small but fixed yields (for some positive constant $c_3 > 0$ and some large constant $C > 0$)

$$\begin{aligned}\|\rho_{-s}(P-z)u\|^2 \geqslant c_3\|\rho_{-s}(P-z)(1-\Gamma)u\|^2 - C\|[P,\Gamma]u\|^2 \\ + \sum_{j=1}^{N} \left(c_3\|\rho_{-s}(P-z)\Gamma_j u\|^2 - C\|[P,\Gamma_j]u\|^2\right).\end{aligned} \tag{A.2}$$

We are now in a position to apply Lemma A.1 to each of the commutator terms. Let $\Gamma_\star$ be any of the microlocal cutoffs in (A.2). The commutator $[P,\Gamma_\star]$ is of order h and supported in a region where every H_p trajectory flows out to $\pm\infty$ in space (in at least one direction). We apply Lemma A.1 with

$$A = \rho_s(1-\Gamma),$$

the constant $C_2 = M$. For $\tilde{B}$, we choose an appropriate microlocal cutoff $\psi_\star$ for each j (and for Γ) so that $\tilde{B} = \tilde{\chi}\psi_\star$ is supported where $H_p \neq 0$ on the flowout of the support of the symbol of $[P,\Gamma_\star]$ and we get

$$\|[P,\Gamma_\star]u\| \leqslant C_M\|(P-z)u\| + \frac{C_0 h}{\sqrt{M}}\|\rho_s(1-\Gamma)u\| + Ch^2\|\tilde{\chi}u\|,$$

with $C_0 > 0$ independent of h and M (of course C_M does depend on M but that is okay for our applications). Plugging into (A.2), we get

$$\begin{aligned}&\|\rho_{-s}(P-z)u\|^2 \\ &\geqslant c_3\|\rho_{-s}(P-z)(1-\Gamma)u\|^2 \\ &\quad + \sum_{j=1}^{N} c_3\|\rho_{-s}(P-z)\Gamma_j u\|^2 \\ &\quad - C'(C_M\|(P-z)u\|^2 + \frac{C_0 h^2}{M}\|\rho_s(1-\Gamma)u\|^2 + Ch^4\sum_{j=0}^{N}\|\tilde{\chi}\psi_j u\|^2).\end{aligned} \tag{A.3}$$

Here the constant $C' > 0$ can be quite large as we are summing over all the commutator terms, but is independent of the parameter M. The sum is from $j = 0$ to N, where we identify $\Gamma_0 = \Gamma$.

For the applications in this paper, the error term $Ch^4\|\tilde{\chi}u\|^2$ is just barely too big, so we apply Lemma A.1 one more time to this term with the same A and $\tilde{B} = \tilde{\chi}_2$ to get

$$h^2\|\tilde{\chi}\psi u\| \leqslant C_M h\|(P-z)u\| + C_M h^2\|\rho_s(1-\Gamma)u\| + Ch^3\|\tilde{\chi}_2 u\|,$$

where now all the constants may be large. Note if the $\alpha_j(h) = \mathcal{O}(h^{-N})$ for a much larger N, then we could apply this argument a finite number of times to further reduce the size of the error. Plugging into (A.3) and absorbing terms which are small into the larger ones, we get

$$\begin{aligned}
&\|\rho_{-s}(P-z)u\|^2 \\
&\quad \geqslant c_3\|\rho_{-s}(P-z)(1-\Gamma)u\|^2 \\
&\qquad + \sum_{j=1}^{N} c_3\|\rho_{-s}(P-z)\Gamma_j u\|^2 \\
&\qquad\quad - C'(C_M\|(P-z)u\|^2 + \frac{C_0h^2}{M}\|\rho_s(1-\Gamma)u\|^2 + Ch^6\|\tilde{\chi}_2 u\|^2).
\end{aligned}$$

Finally, we move the negative terms with $(P-z)$ to the left hand side and apply all the assumed black box microlocal estimates to conclude

$$\begin{aligned}
&\|\rho_{-s}(P-z)u\|^2 \\
&\quad \geqslant c_3h^2\|\rho_s(1-\Gamma)u\|^2 \\
&\qquad + \sum_{j=1}^{N} c_3\frac{h^2}{\alpha_j^2(h)}\|\Gamma_j u\|^2 \\
&\qquad\quad - C'(\frac{C_0h^2}{M}\|\rho_s(1-\Gamma)u\|^2 + Ch^6\|\tilde{\chi}_2 u\|^2).
\end{aligned}$$

By taking $M > 0$ sufficiently large, the term

$$-C'\frac{C_0h^2}{M}\|\rho_s(1-\Gamma)u\|^2$$

can be absorbed into the positive term with the same cutoffs. After taking the worst lower bound and summing over our partition of unity, this gives

$$\begin{aligned}
&\|\rho_{-s}(P-z)u\|^2 \\
&\quad \geqslant c_4h^2\min\left\{1, \alpha_1^{-2}(h), \dots, \alpha_N^{-2}(h)\right\}\|\rho_s u\|^2 - C''h^6\|\tilde{\chi}_2 u\|^2.
\end{aligned}$$

We finish by observing that

$$h^6 \ll h^2\min\left\{1, \alpha_1^{-2}(h), \dots, \alpha_N^{-2}(h)\right\},$$

and $\rho_s \equiv 1$ on $\operatorname{supp}\tilde{\chi}_2$, so this term can be absorbed as well, leaving us with a final estimate of

$$\begin{aligned}
&\|\rho_{-s}(P-z)u\|^2 \\
&\quad \geqslant c_4h^2\min\left\{1, \alpha_1^{-2}(h), \dots, \alpha_N^{-2}(h)\right\}\|\rho_s u\|^2.
\end{aligned}$$

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